Introduction to quantum Gaussian systems

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Abstract

Quantum Gaussian systems provide the mathematical model for the electromagnetic radiation in the quantum regime. Quantum Gaussian channels are the physical operation on quantum Gaussian systems that model the propagation of electromagnetic signals through optical fibers, which are the main mean to distribute quantum states in quantum key distribution and in the forthcoming quantum internet. Quantum Gaussian states are the most relevant states of quantum Gaussian system, since they can be easily prepared experimentally and are the best codewords to communicate through quantum Gaussian channels.

This lecture will provide an introduction to quantum information with quantum Gaussian systems. The topics will include the canonical commutation relations and quantum Gaussian states, channels and measurements. Particular emphasis will be put on coherent, squeezed and thermal states, quantum Gaussian attenuator and amplifier channels and homodyne and heterodyne measurements.

- Electromagnetic field main information carrier
- Mathematical model: ensemble of harmonic oscillators (quantum Gaussian system)
- Restricting to finite number of modes, $\mathcal{H} = L^2(\mathbb{R}^m)$ wavefunctions of m harmonic oscillators
- Quadratures $Q_1 \ldots Q_m$, $P_1 \ldots P_m$ put together in

$$R_1 = Q_1,\, R_2 = P_1,\, \ldots,\, R_{2m-1} = Q_m,\, R_{2m} = P_m$$

- Canonical commutation relations and symplectic form

$$[Q_j,\, P_k] = \mathrm{i}\,\delta_{jk}\,\mathbb{I} \qquad (\hbar = 1)$$
$$[R_j,\, R_k] = \mathrm{i}\,\Delta_{jk}\,\mathbb{I} \qquad \Delta = \bigoplus_{i=1}^{m} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Homodyne measurement: measure $\sum_{i=1}^{2m} c_i\, R_i$

- First moments and covariance matrix

$$r_i = \mathrm{Tr}\,[R_i\,\rho] \qquad \sigma_{ij} = \frac{1}{2}\,\mathrm{Tr}\,[\{R_i - r_i\,\mathbb{I},\ R_j - r_j\,\mathbb{I}\}\,\rho]$$

- Uncertainty principle

$$R(z) = \sum_{i=1}^{2m} z_i\,(R_i - r_i\,\mathbb{I}) \qquad z \in \mathbb{C}^{2m}$$

$$\mathrm{Tr}\left[R(z)^\dagger R(z)\,\rho\right] \geq 0 \qquad \Longrightarrow \qquad \sigma \geq \pm\frac{\mathrm{i}}{2}\,\Delta \qquad \det\sigma \geq 1 \text{ for } m = 1$$

- Ladder operators

$$a_j = \frac{Q_j + \mathrm{i}\,P_j}{\sqrt{2}} \qquad \left[a_j,\, a_k^\dagger\right] = \delta_{jk}\,\mathbb{I} \qquad \alpha_j = \frac{q_j + \mathrm{i}\,p_j}{\sqrt{2}}$$

- Photon-number Hamiltonian ($\omega = 1$)

$$H = \sum_{i=1}^{m} a_i^\dagger a_i = \sum_{i=1}^{m} \frac{Q_i^2 + P_i^2 - \mathbb{I}}{2} \qquad \mathrm{Tr}\,[H\,\rho] = \frac{\|r\|^2 + \mathrm{tr}\,\sigma - m}{2}$$

- Vacuum and Fock states ($m = 1$)

$$a|0\rangle = H|0\rangle = 0 \qquad |n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \qquad H|n\rangle = n|n\rangle \qquad n \in \mathbb{N}$$

- Unitary operators with simple action on quadratures?
- Displacement operators

$$D(\alpha) = \exp\left(\sum_{i=1}^{m}\left(\alpha_i\,a_i^\dagger - \alpha_i^*\,a_i\right)\right) \qquad D(\alpha)^\dagger\,a_i\,D(\alpha) = a_i + \alpha\,\mathbb{I} \qquad \alpha \in \mathbb{C}^m$$

- Symplectic group

$$\mathrm{Sp}(2m,\mathbb{R}) = \left\{S \in \mathbb{R}^{2m\times 2m} : S\,\Delta\,S^T = \Delta\right\} \qquad \mathrm{Sp}(2,\mathbb{R}) = \left\{S \in \mathbb{R}^{2\times 2} : \det S = 1\right\}$$

- Symplectic unitaries: symplectic transformations of quadratures

$$U(S)^\dagger\,R_i\,U(S) = \sum_{j=1}^{2m} S_{ij}\,R_j \qquad U(S)\,U(S') = U(S\,S') \qquad \sigma \mapsto S\,\sigma\,S^T$$

- Passive symplectic unitaries (Exercise)

$$U(S)^\dagger\, H\, U(S) = H \qquad S^T S = I_{2m} \Longrightarrow S \in \mathrm{Sp}(2m,\mathbb{R}) \cap \mathrm{O}(2m,\mathbb{R}) \simeq \mathrm{U}(m)$$

$$U(S)^\dagger\, a_i\, U(S) = \sum_{j=1}^m \mathcal{U}(S)_{ij}\, a_j \qquad \mathcal{U}(S) \in \mathrm{U}(m) \qquad U(S)|0\rangle = |0\rangle$$

  - Phase shifter ($m = 1$)

$$S(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad t \in \mathbb{R} \qquad U(t) = \mathrm{e}^{-\mathrm{i}\,H\,t}$$

  - Beam-splitter ($m = 2$)

$$S(\eta) = \begin{pmatrix} \sqrt{\eta}\, I_2 & -\sqrt{1-\eta}\, I_2 \\ \sqrt{1-\eta}\, I_2 & \sqrt{\eta}\, I_2 \end{pmatrix} \qquad 0 \le \eta \le 1$$
$$U(\eta) = \exp\left(\arccos\sqrt{\eta}\,\left(a^\dagger\, b - b^\dagger\, a\right)\right)$$

- Active symplectic unitaries

  - One-mode squeezing ($m = 1$)

$$S(k) = \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix} \qquad U(k) = \exp\left(\ln k\, \frac{{a^\dagger}^2 - a^2}{2}\right) \qquad k > 0$$

  - Two-mode squeezing ($m = 2$)

$$S(\kappa) = \begin{pmatrix} \sqrt{\kappa}\, I_2 & \sqrt{\kappa - 1}\, \sigma_Z \\ \sqrt{\kappa-1}\, \sigma_Z & \sqrt{\kappa}\, I_2 \end{pmatrix} \qquad \kappa \ge 1 \qquad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$U(\kappa) = \exp\left(\mathrm{arccosh}\sqrt{\kappa}\,\left(a^\dagger\, b^\dagger - a\, b\right)\right)$$

- Phase shifters, beam-splitters and one-mode squeezers generate all symplectic unitaries: any $S \in \mathrm{Sp}(2m,\mathbb{R})$ can be decomposed as

$$S = O_1 \left(\bigoplus_{i=1}^m S(k_i)\right) O_2 \qquad O_1,\, O_2 \in \mathrm{Sp}(2m,\mathbb{R}) \cap \mathrm{O}(2m,\mathbb{R})$$

- Quantum Gaussian states: thermal states of quadratic Hamiltonians, completely determined by first and second moments

$$\omega \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{2m} (R_i - r_i)\, h_{ij}\, (R_j - r_j)\right) \qquad r \in \mathbb{R}^{2m},\ h \in \mathbb{R}^{2m\times 2m},\ h > 0$$

– Outcome of heterodyne measurement has Gaussian distribution given by r, σ

– Thermal quantum Gaussian states ($m = 1$)

$$\omega(E) = \frac{1}{E+1}\sum_{n=0}^{\infty}\left(\frac{E}{E+1}\right)^n |n\rangle\langle n| \qquad E \geq 0$$

$$r = 0 \qquad \sigma = \left(E + \tfrac{1}{2}\right) I_2 \qquad h = \ln\frac{E+1}{E}\, I_2$$

$$\omega(0) = |0\rangle\langle 0|$$

$$S(\omega(E)) = (E+1)\ln(E+1) - E\ln E := g(E)$$

– Coherent states ($m = 1$)

$$|\alpha\rangle = D(\alpha)|0\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \alpha \in \mathbb{C} \qquad \sigma = \tfrac{1}{2} I_2 \qquad a|\alpha\rangle = \alpha|\alpha\rangle$$

$$U(\eta)|\alpha,\beta\rangle = \left|\sqrt{\eta}\,\alpha - \sqrt{1-\eta}\,\beta,\ \sqrt{1-\eta}\,\alpha + \sqrt{\eta}\,\beta\right\rangle$$

– One-mode squeezed vacuum states ($m = 1$)

$$|\phi_k\rangle = U(S(k))|0\rangle = \sqrt{\frac{2k}{k^2+1}}\sum_{n=0}^{\infty}\left(\frac{k^2-1}{2\left(k^2+1\right)}\right)^n \frac{\sqrt{(2n)!}}{n!}|2n\rangle$$

$$\sigma = \begin{pmatrix} \frac{k^2}{2} & 0 \\ 0 & \frac{1}{2k^2} \end{pmatrix}$$

– Two-mode squeezed vacuum states ($m = 2$)

$$|\phi_\kappa\rangle = U(S(\kappa))|0\rangle = \frac{1}{\sqrt{\kappa}}\sum_{n=0}^{\infty}\left(\frac{\kappa-1}{\kappa}\right)^{\frac{n}{2}}|n\rangle\otimes|n\rangle$$

$$\sigma = \begin{pmatrix} \left(\kappa - \frac{1}{2}\right) I_2 & \sqrt{\kappa\left(\kappa-1\right)}\,\sigma_Z \\ \sqrt{\kappa\left(\kappa-1\right)}\,\sigma_Z & \left(\kappa - \frac{1}{2}\right) I_2 \end{pmatrix}$$

– Normal form of quantum Gaussian states

$$\sigma = S\left(\bigoplus_{i=1}^{m}\nu_i\, I_2\right)S^T \qquad \omega = D(\alpha)\,U(S)\left(\bigotimes_{i=1}^{m}\omega(\nu_i - \tfrac{1}{2})\right)U(S)^\dagger\, D(\alpha)^\dagger$$

– $\nu_i \geq \frac{1}{2}$: symplectic eigenvalues of σ

- Quantum Gaussian states maximize von Neumann entropy for given first and second moments: $\forall$ quantum state ρ, let ρ_G be the Gaussian states with same first and second moments. Then,

$$S(\rho\|\rho_G) = S(\rho_G) - S(\rho) \geq 0$$

- Heterodyne measurement: send ρ through $\eta = 1/2$ beam-splitter, then measure Q on first port and P on second (Exercise)

$$\alpha = q + \mathrm{i}\, p \qquad \mathrm{d}p(\alpha) = \langle\alpha|\rho|\alpha\rangle\, \frac{\mathrm{d}\alpha}{\pi^m}$$

- Quantum Gaussian channels: preserve set of quantum Gaussian states

$$\Phi(\rho) = \mathrm{Tr}_B\left[U(S)\,(\rho\otimes\omega)\,U(S)^\dagger\right]$$
$$r \mapsto K^T\, r \qquad \sigma \mapsto K^T\,\sigma\,K + \alpha \qquad \alpha \geq \pm\frac{\mathrm{i}}{2}\left(K^T\,\Delta\,K - \Delta\right)$$

  - Gauge-covariant if commuting with time evolution

$$\Phi\left(\mathrm{e}^{-\mathrm{i}\,H\,t}\,\rho\,\mathrm{e}^{\mathrm{i}\,H\,t}\right) = \mathrm{e}^{-\mathrm{i}\,H\,t}\,\Phi(\rho)\,\mathrm{e}^{-\mathrm{i}\,H\,t} \qquad K^T\,S(t) = S(t)\,K^T \qquad S(t)\,\alpha\,S(t)^T = \alpha$$

  - Quantum Gaussian attenuator $\mathcal{E}_{\eta,E}$: $S = \bigoplus_{i=1}^m S(\eta)$ (beam-splitter), $\omega = \omega(E)^{\otimes m}$; noiseless for $E = 0$. Models signal propagation through optical fibers and free space.

$$r \mapsto \sqrt{\eta}\, r \qquad \sigma \mapsto \eta\,\sigma + (1-\eta)\left(E + \tfrac{1}{2}\right) I_{2m}$$
$$\mathcal{E}_{\eta,0}(|\alpha\rangle\langle\alpha|) = |\sqrt{\eta}\,\alpha\rangle\,\langle\sqrt{\eta}\,\alpha|$$

  - Quantum Gaussian amplifier $\mathcal{A}_{\kappa,E}$: $S = \bigoplus_{i=1}^m S(\kappa)$ (two-mode squeezing), $\omega = \omega(E)^{\otimes m}$; noiseless for $E = 0$. Models signal amplification.

$$r \mapsto \sqrt{\kappa}\, r \qquad \sigma \mapsto \kappa\,\sigma + (\kappa - 1)\left(E + \tfrac{1}{2}\right) I_{2m}$$

  - $m = 1$: noiseless quantum Gaussian attenuators and amplifiers generate gauge-covariant quantum Gaussian channels

References

[1] Alessio Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods*. CRC Press, 2017.

[2] Alexander Semenovich Holevo. *Quantum Systems, Channels, Information: A Mathematical Introduction*. De Gruyter Studies in Mathematical Physics. De Gruyter, 2013.