Introduction to contextuality

Victoria J Wright
Quantum theory inherently probabilistic

Upsetting:
Quantum theory inherently probabilistic

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Upsetting:

Could it be deterministic but we just don’t know everything?
Quantum theory inherently probabilistic

Upsetting:

Could it be deterministic but we just don’t know everything?

\[ |0\rangle \]
Quantum theory inherently probabilistic

Upsetting:

Could it be deterministic but we just don’t know everything?

State

$|0\rangle$

Measurement

$\{1+X+1, 1-X-1\}$

$\frac{1}{2} \quad \frac{1}{2}$
Quantum theory inherently probabilistic

Upsetting:

Could it be deterministic but we just don’t know everything?

\[
\begin{align*}
\text{State} & \quad 10 \rangle \\
\text{Measurement} & \quad \{1+X+1, 1-X-1\} \\
\text{Actually} & \quad \lambda_+ \quad \lambda_- \\
\frac{1}{2} & \quad \frac{1}{2}
\end{align*}
\]
Quantum theory inherently probabilistic

Upsetting:

Could it be deterministic but we just don’t know everything?

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<td>$</td>
<td>0\rangle$</td>
<td>${1+X+1, 1-X-1}$</td>
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<tr>
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Yes!
Quantum theory inherently probabilistic

Upsetting:

Could it be deterministic but we just don’t know everything?

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Yes!

But…then it also must be (measurement) contextual

Observables that are (statistically) indistinguishable to us, are in fact different!
Quantum theory inherently probabilistic

Upsetting:

Could it be deterministic but we just don’t know everything?

State

\[|0\rangle\]

Measurement

\[\{1+X+1, 1-X-1\}\]

\[\frac{1}{2} \quad \frac{1}{2}\]

Actually

\[\lambda_+ \quad \lambda_-\]

\[\frac{1}{2} \quad \frac{1}{2}\]

Yes!

KOCHEN-SPECKER THEOREM

But…then it also must be (measurement) contextual

Observables that are (statistically) indistinguishable to us, are in fact different!
CONTEXTS OF $Z \otimes I$

1. \{100\>, 101\>, 110\>, 111\>\}  
   +1 \quad +1 \quad -1 \quad -1  
   \Rightarrow ZI

2. \{10\>, 10\>, 11\>, 11\>\}  
   +1 \quad +1 \quad -1 \quad -1  
   \Rightarrow ZI

Two different contexts for the same measurement:
- Context: which other observables I simultaneously measure
- OR which experimental procedure I use

Noncontextuality:

$\emptyset \quad ZI \quad = \quad \emptyset \quad ZI$

fundamentally same observable
CONTEXTS OF $Z \otimes I$

1. $\left\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \right\}$
   $\text{ZI}$ $\begin{array}{cccc}
   +1 & +1 & -1 & -1 \\
   +1 & -1 & +1 & -1 \\
   +1 & -1 & -1 & +1
   \end{array}$

2. $\left\{ |0+\rangle, |0-\rangle, |1+\rangle, |1-\rangle \right\}$
   $\text{ZI}$ $\begin{array}{cccc}
   +1 & +1 & -1 & -1
   \end{array}$

Two different contexts for the same measurement:
- Context: which other observables I simultaneously measure
- OR which experimental procedure I use

Noncontextuality:

$\text{ZI}_1 = \text{ZI}_2$

fundamentally same observable
Two different contexts for the same measurement
- Context: which other observables I simultaneously measure
- OR which experimental procedure I use

Noncontextuality:

\[ 1 \text{ ZI} = 2 \text{ ZI} \]

fundamentally same observable
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

Epistemic states

Λ

μ

ρ
**Ontic states**

- **SPEC** \( \lambda(A) \in \sigma(A) \)
- **FUNC** \( \lambda(f(A)) = f(\lambda(A)) \)

**Epistemic states**

\[
P(A=a | \rho) = \int \mu_\rho(\lambda) \, d\lambda \quad \text{such that} \quad \lambda(A) = a
\]
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

[SPEC] $\lambda(A) \in \sigma(A)$

[FUNC] $\lambda(f(A)) = f(\lambda(A))$

E.g. $\lambda(A^2) = [\lambda(A)]^2$

Epistemic states

$P(A=a | \rho) = \int \mu_{\rho}(\lambda) d\lambda$

$\lambda$ s.t. $\lambda(A) = a$

probability measure on $\Lambda$
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

[SPEC] \( \lambda(A) \in \sigma(A) \)

[FUNC] \( \lambda(f(A)) = f(\lambda(A)) \)

Epistemic states

\[
P(A=a \mid \rho) = \int \mu_\rho(A) d\lambda
\]

\( \lambda \text{ s.t. } \lambda(A) = a \)

probability measure on \( \Lambda \)
Ontic states

[SPEC] \( \lambda(A) \in \sigma(A) \)

[FUNC] \( \lambda(f(A)) = f(\lambda(A)) \)

Epistemic states

\[
P(A = a \mid \rho) = \int \mu_\rho(\lambda) d\lambda
\]

\( \lambda \text{ s.t. } \lambda(A) = a \)

probability measure on \( \Lambda \)
**Ontic states**

[SPEC] $\lambda(A) \in \sigma(A)$

[FUNC] $\lambda(f(A)) = f(\lambda(A))$

**Epistemic states**

$$P(A=a \mid \rho) = \int \mu_{\rho}(\lambda) d\lambda$$

where $\lambda$ is such that $\lambda(A) = a$.

**ZI** ZI ZZ I

**IX** XI XX I

**ZX** XZ YY I
**Ontic states**

- **SPEC** \( \lambda(A) \in \sigma(A) \)
- **FUNC** \( \lambda(f(A)) = f(\lambda(A)) \)

**Epistemic states**

\[
P(A=a \mid \rho) = \int \mu_\rho(\lambda) d\lambda \\
\text{subject to} \lambda(A)=a
\]
**Ontic states**

[SPEC] \( \lambda(A) \in \sigma(A) \)

[FUNC] \( \lambda(f(A)) = f(\lambda(A)) \)

\[ \lambda(AB) = \lambda(A) \lambda(B) \text{ if } [A, B] = 0 \]

**Epistemic states**

\[ P(\lambda = a | \rho) = \int \mu_{\rho}(\lambda) d\lambda \]

\( \lambda \text{ s.t. } \lambda(A) = a \)

\( \mu_{\rho} \) is a probability measure on \( \Lambda \)
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

[SPEC] \( \lambda(A) \in \sigma(A) \)

[FUNC] \( \lambda(f(A)) = f(\lambda(A)) \)

Epistemic states

\[
P(A=a | \rho) = \int \mu_\rho(A) d\lambda
\]

Exercise!

Hint: \( [A, B] = 0 \implies \exists C \text{ such that } A = f(C) \text{ and } B = g(C) \)

\( \lambda(AB) = \lambda(A)\lambda(B) \text{ if } [A, B] = 0 \)

\( \lambda \text{ s.t. } \lambda(A) = a \)

probability measure on \( \Lambda \)
Exercise!

Hint: $[A, B] = 0 \implies \exists C$ such that $A = f(C)$ and $B = g(C)$

Probability measure on $\Lambda$

$P(A=a \mid \rho) = \int \mu_\rho(A) d\lambda$

$\lambda$ s.t. $\lambda(A) = a$

$\lambda(AB) = \lambda(A)\lambda(B) \text{ if } [A, B] = 0$
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

[SPEC] \( \lambda(A) \in \sigma(A) \)

[FUNC] \( \lambda(f(A)) = f(\lambda(A)) \)

\( \lambda(AB) = \lambda(A)\lambda(B) \) if \( [A,B] = 0 \)

Exercise!

Hint: \( [A, B] = 0 \implies \exists C \) such that \( A = f(C) \) and \( B = g(C) \)

Epistemic states

\( P(A=a|\rho) = \int \mu_{\rho}(\Lambda) d\lambda \)

\( \lambda \) s.t. \( \lambda(A) = a \)

probability measure on \( \Lambda \)

\( \lambda(zz)\lambda(iz)\lambda(zx)\lambda(zi)\lambda(xz)\lambda(xz)\lambda(zy)\lambda(zy) \)
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

- [SPEC] $\lambda(A) \in \sigma(A)$
- [FUNC] $\lambda(f(A)) = f(\lambda(A))$

Epistemic states

$P(A=a \mid \rho) = \int \mu_{\rho}(A) d\lambda$

Exercise!

Hint: $[A, B] = 0 \implies \exists C$ such that $A = f(C)$ and $B = g(C)$

$\lambda(AB) = \lambda(A)\lambda(B)$ if $[A, B] = 0$

Probability measure on $\Lambda$

$\lambda(zI)\lambda(iz)\lambda(zz)\lambda(ix)\lambda(xi)\lambda(xx)\lambda(zx)\lambda(xz)\lambda(yy)$

$\lambda(I)$
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

Ontic states

- **[SPEC]** \( \lambda(A) \in \sigma(A) \)
- **[FUNC]** \( \lambda(f(A)) = f(\lambda(A)) \)

Epistemic states

\[ P(A=a \mid \rho) = \int \mu_\rho(\lambda) d\lambda \]

\( \lambda \) s.t. \( \lambda(A) = a \)

Exercise!
Hint: \( [A, B] = 0 \implies \exists C \) such that \( A = f(C) \) and \( B = g(C) \)

\( \lambda(AB) = \lambda(A)\lambda(B) \) if \( [A, B] = 0 \)

\[ \lambda \begin{pmatrix} ZI & IZ & ZZ & I \\ IX & XI & XX & I \\ ZX & XZ & YY & I \end{pmatrix} = 1 \]
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<td>( P(A=a \mid \rho) = \int \mu_\rho(A) d\lambda )</td>
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<td>[FUNC] ( \lambda(f(A)) = f(\lambda(A)) )</td>
<td>( \lambda \text{ s.t. } \lambda(A) = a )</td>
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Exercise!

Hint: \([A, B] = 0 \implies \exists C \) such that \( A = f(C) \) and \( B = g(C) \)

\[ \lambda(AB) = \lambda(A)\lambda(B) \text{ if } [A, B] = 0 \]

\[ \begin{align*}
\lambda(I) & \equiv 1 \\
\lambda(I) & \equiv 1 \\
\lambda(I) & \equiv 1 \\
\lambda(I) & \equiv 1 \\
\lambda(I) & \equiv 1 \\
\lambda(I) & \equiv 1 \\
\lambda(I) & \equiv 1 \\
\end{align*} \]
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

**Ontic states**

- [SPEC] $\lambda(A) \in \sigma(A)$
- [FUNC] $\lambda(f(A)) = f(\lambda(A))$

**Epistemic states**

$P(A=a \mid \rho) = \int \mu_\rho(\lambda) d\lambda$

$\lambda$ s.t. $\lambda(A) = a$

Exercise!

Hint: $[A, B] = 0 \implies \exists C$ such that $A = f(C)$ and $B = g(C)$

$\lambda(AB) = \lambda(A)\lambda(B)$ if $[A, B] = 0$
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

**Ontic states**

- **SPEC** \( \lambda(A) \in \sigma(A) \)
- **FUNC** \( \lambda(f(A)) = f(\lambda(A)) \)

**Epistemic states**

\[
P(A=a | \rho) = \int \mu_{\rho}(\lambda) d\lambda
\]

\( \lambda \) s.t. \( \lambda(A) = a \)

Exercise!

**Hint:** \([A, B] = 0 \implies \exists C\) such that

\( A = f(C) \) and \( B = g(C) \)

\( \lambda(AB) = \lambda(A)\lambda(B) \) if \( [A, B] = 0 \)

**Example**

\[
\begin{align*}
\lambda(I) & = 1 \\
\lambda(I) & = 1 \\
\lambda(I) & = 1 \\
\lambda(I) & = -1
\end{align*}
\]
DETERMINISTIC NONCONTEXTUAL ONTOLOGICAL MODELS

**Ontic states**

[SPEC] \( \lambda(A) \in \sigma(A) \)

[FUNC] \( \lambda(f(A)) = f(\lambda(A)) \)

**Epistemic states**

\[
P(A=a | \rho) = \int \lambda \ s.t. \lambda(A) = a \\rho(\lambda) \ d\lambda
\]

**Exercise!**

Hint: \([A, B] = 0 \implies \exists C \) such that \( A = f(C) \) and \( B = g(C) \)

\[ \lambda(AB) = \lambda(A)\lambda(B) \text{ if } [A, B] = 0 \]

\[
\begin{align*}
\lambda(II) \lambda(IZ) \lambda(IZ) \lambda(IX) \lambda(XI) \lambda(XX) & \lambda(ZX) \lambda(XZ) \lambda(YY) \\
\lambda(I) \quad \lambda(I) \quad \lambda(I) \quad \lambda(I) \quad \lambda(I) \quad \lambda(-I) \\
\lambda(IZ) \lambda(I) \lambda(Z) & \lambda(X) \lambda(Z) \lambda(X) \\
\lambda(XI) \lambda(I) \lambda(X) & \lambda(Z) \lambda(X) \lambda(Z) \\
\lambda(X) \lambda(II) & \lambda(Z) \lambda(X) \lambda(Z) \\
\lambda(ZI) & \lambda(Z) \lambda(X) \lambda(Z) \\
\lambda(Z) & \lambda(X) \lambda(Z) \\
\lambda & \lambda(-I)
\end{align*}
\]
No deterministic noncontextual ontic states

**KOCHEN-SPECKER THEOREM**

there is no deterministic noncontextual ontological model for QT

*K. In dimensions $\geq 3$

No deterministic noncontextual ontic states also follows from Gleason’s theorem [Exercise]

Kochen-Specker Theorem
there is no deterministic noncontextual ontological model for QT

No deterministic noncontextual ontic states

also follows from Gleason’s theorem [Exercise]

KOCHEN-SPECKER THEOREM

there is no deterministic noncontextual ontological model for QT

ontic states + epistemic states


Leifer, arXiv:1409.1570

* in dimensions $\geq 3$
\[
\{|10\rangle, |11\rangle, |01\rangle, |00\rangle\} \\
\{|00\rangle, \, |11\rangle, \, |10\rangle, \, |01\rangle\} \\
\{|0\rangle, |1\rangle, |\pm\rangle\} \\
\{|0\rangle, |1\rangle, |\Phi_+\rangle, |\Phi_-\rangle\} \\
\{|\alpha_+\rangle, |\alpha_-\rangle, |\beta_+\rangle, |\beta_-\rangle\} \\
\{|0\rangle, |1\rangle, |\psi_+\rangle, |\psi_-\rangle\}
\]

\[
\{|10\rangle, |0\rangle, |1\rangle, |\pm\rangle\} \\
\{|0\rangle, |1\rangle, |\Phi_+\rangle, |\Phi_-\rangle\} \\
\{|0\rangle, |1\rangle, |\alpha_+\rangle, |\alpha_-\rangle\} \\
\{|0\rangle, |1\rangle, |\beta_+\rangle, |\beta_-\rangle\} \\
\{|0\rangle, |1\rangle, |\psi_+\rangle, |\psi_-\rangle\}
\]

\[\text{[SPEC]} \quad \lambda(A) \in \sigma(A)\]

\[\text{[FUNC]} \quad \lambda(f(A)) = f(\lambda(A))\]
\[
\begin{align*}
\{ |00\>, |10\>, |11\>, |11\> \} & \quad \{ |00\>, |10\>, |11\>, |11\> \} \quad \{ |00\>, |10\>, |11\>, |11\> \} \\
+1 & +1 -1 -1 \quad \pm \quad +1 & +1 -1 -1 \quad \pm \\
+1 & +1 -1 -1 \quad \pm \quad +1 & +1 -1 -1 \quad \pm \\
+1 & +1 -1 -1 \quad \pm \quad +1 & +1 -1 -1 \quad \pm \\
\{ |\alpha\>, |\alpha\>, |\beta\>, |\beta\> \} & \quad \{ |\alpha\>, |\alpha\>, |\beta\>, |\beta\> \} \\
+1 & +1 -1 -1 \quad \pm \quad +1 & +1 -1 -1 \quad \pm \\
+1 & +1 -1 -1 \quad \pm \quad +1 & +1 -1 -1 \quad \pm \\
+1 & +1 -1 -1 \quad \pm \quad +1 & +1 -1 -1 \quad \pm \\
|\Phi_\pm\> = |00\> \pm |11\> & \quad |\Psi_\pm\> = |01\> \pm |10\> \\
|\alpha_\pm\> = |0+\> \pm |1-\> & \quad |\beta_\pm\> = |1+\> \pm |0-\>
\end{align*}
\]

SPEC \[\lambda(A) \in \sigma(A)\]

FUNC \[\lambda(f(A)) = f(\lambda(A))\]
KS-colouring: assign 0 or 1 to each ray such that every hyperedge contains exactly one 1.
Observables vs measurement outcomes

KS-colouring: assign 0 or 1 to each ray such that every hyperedge contains exactly one 1

\[ |\Phi_{\pm}\rangle = |00\rangle \pm |11\rangle \]
\[ |\Psi_{\pm}\rangle = |01\rangle \pm |10\rangle \]
\[ |\alpha_{\pm}\rangle = |0+\rangle \pm |1-\rangle \]
\[ |\beta_{\pm}\rangle = |1+\rangle \pm |0-\rangle \]
\[ |\Phi_\pm\rangle = |00\rangle \pm |11\rangle \]
\[ |\Psi_\pm\rangle = |01\rangle \pm |10\rangle \]
\[ |\alpha_\pm\rangle = |0+\rangle \pm |1-\rangle \]
\[ |\beta_\pm\rangle = |1+\rangle \pm |0-\rangle \]

Cabello, Estebananz, García-Alcaine
\begin{align*}
\Phi_+ &= |00\rangle \pm |11\rangle \\
\Psi_- &= |01\rangle \pm |10\rangle \\
\alpha_+ &= |0^+\rangle \pm |1^-angle \\
\beta_- &= |1^+\rangle \pm |0^-\rangle \\
\end{align*}

\[ v(|11\rangle) + v(|10\rangle) + v(|0^+\rangle) + v(|0^\rangle) = 1 \]
\[ \psi(11) + \psi(10) + \psi(0+) + \psi(0-) = 1 \]

\[ Cabello, Estebananz, García-Alcaine \]

|Φ⟩ ± |⟩ = |00⟩ ± |11⟩
|Ψ⟩ ± |⟩ = |01⟩ ± |10⟩
|β⟩ ± |⟩ = |1⟩ ± |0⟩
|α⟩ ± |⟩ = |0⟩ ± |1⟩

State independent proofs of KS theorem
No deterministic noncontextual ontic states

Observables

\(Z \otimes I\)  \(I \otimes Z\)  \(Z \otimes Z\)
\(I \otimes X\)  \(X \otimes I\)  \(X \otimes X\)
\(Z \otimes X\)  \(X \otimes Z\)  \(Y \otimes Y\)
State independent proofs of KS theorem
No deterministic noncontextual ontic states

|Φ±⟩ = |00⟩ ± |11⟩
|Ψ±⟩ = |01⟩ ± |10⟩
|α±⟩ = |0+⟩ ± |1−⟩
|β±⟩ = |1+⟩ ± |0−⟩

Measurement outcomes

Observables

Z ⊗ I  I ⊗ Z  Z ⊗ Z
I ⊗ X  X ⊗ I  X ⊗ X
Z ⊗ X  X ⊗ Z  Y ⊗ Y
State dependent proofs of KS theorem

Maybe deterministic noncontextual ontic states

BUT
do not mix together to reproduce quantum statistics

\(|\Phi_\pm\rangle = |00\rangle \pm |11\rangle\)
\(|\Psi_\pm\rangle = |01\rangle \pm |10\rangle\)
\(|a_\pm\rangle = |0+\rangle \pm |1-\rangle\)
\(|\beta_\pm\rangle = |1+\rangle \pm |0-\rangle\)

Cabello, Estebananz, García-Alcaine
State dependent proofs of KS theorem

Maybe deterministic noncontextual ontic states

BUT
do not mix together to reproduce quantum statistics

\[ \Phi^\pm = |00\rangle \pm |11\rangle \]
\[ \Psi^\pm = |01\rangle \pm |10\rangle \]
\[ \alpha^\pm = |0+\rangle \pm |1-\rangle \]
\[ \beta^\pm = |1+\rangle \pm |0-\rangle \]

Cabello, Estebananz, García-Alcaine
State dependent proofs of KS theorem

Maybe deterministic noncontextual ontic states

BUT
do not mix together to reproduce quantum statistics

Cabello, Estebaran, García-Alcaine
State: \( |\Psi_-\rangle = |01\rangle - |10\rangle \)

\[
\begin{align*}
|\Phi_-angle & = |00\rangle \pm |11\rangle \\
|\Psi_\pm\rangle & = |01\rangle \pm |10\rangle \\
|\alpha_\pm\rangle & = |0+\rangle \pm |1-\rangle \\
|\beta_\pm\rangle & = |1+\rangle \pm |0-\rangle \\
\end{align*}
\]

State dependent proofs of KS theorem

Maybe deterministic noncontextual ontic states

BUT
do not mix together to reproduce quantum statistics

Cabello, Estebaranz, García-Alcaine

State: $|\Psi_\pm\rangle = |01\rangle - |10\rangle$

State dependent proofs of KS theorem

Maybe deterministic noncontextual ontic states

BUT
do not mix together to reproduce quantum statistics

Cabello, Estebananz, García-Alcaine
State: $|\Psi_-\rangle = |01\rangle - |10\rangle$

Exercise! Hint: think about the zeroes (you don’t have to write out all the colourings)
State: $|\Psi_-\rangle = |01\rangle - |10\rangle$

Also, noncontextuality inequalities e.g. Bell inequalities
Acín, Fritz, Leverrier, Sainz arXiv:1212.4084

Exercise! Hint: think about the zeroes (you don’t have to write out all the colourings)

Cabello, Estebananz, García-Alcaíne
Kochen-Specker

DETERMINISM + MEASUREMENT NONCONTEXTUALITY

Difference between quantum and classical physics

Related to other forms of "nonclassicality"

Nonlocality:

- noncontextuality motivated via locality
- Bell inequality violations are state dependent proofs

Requires incompatibility
Spekkens

DETERMINISM + MEASUREMENT NONCONTEXTUALITY

Preparation noncontextuality ( + transformation noncontextuality )

Spekkens

DETERMINISM + MEASUREMENT NONCONTEXTUXTUALITY

Preparation noncontextuality (+ transformation noncontextuality)

DIFFERENCES WITH KOCHEN-SPECKER

POVMs not just PVMs

Dimension 2

Makes sense for general operational theories

Coincides with classicality in general probabilistic theories, i.e., simplex-embeddability arXiv:1911.10386v2
Preparation contexts: Different ways of preparing a system that cannot be distinguished by any measurement.

- Preparation context $P$: $\frac{1}{2}\left|0\right\rangle + \frac{1}{2}\left|1\right\rangle$
- Preparation context $P'$: $\frac{1}{2}\left|\pm\right\rangle$

The diagrams illustrate the superposition of states, with $\left|0\right\rangle$ and $\left|1\right\rangle$ representing the two states, and $\left|\pm\right\rangle$ representing the superposition.
GENERIC OPERATIONAL THEORY:

\( P, P' \in \mathcal{P} \) are equivalent:

\[ P \sim P' \]

\[ \text{Prob}(k|P, M) = \text{Prob}(k|P', M) \quad \text{for all} \quad [k|M] \in \mathcal{E} \]

\([k|M], [k'|M'] \in \mathcal{E} \) are equivalent:

\[ [k|M] \sim [k'|M'] \]

\[ \text{Prob}(k|P, M) = \text{Prob}(k'|P, M') \quad \text{for all} \quad P \in \mathcal{P} \]
NONCONTEXTUAL ONTOLOGICAL MODEL

\( \Lambda, \Sigma \) ontic state space

\( \mu_P : \Sigma \rightarrow [0,1] \) epistemic state

\( \xi_M(k|\lambda) \) response function

\[
\text{Prob}(k|P,M) = \int_\Lambda \xi_M(k|\lambda)\mu_P(\lambda)d\lambda
\]

\( P \sim P' \implies \mu_P = \mu_{P'} \)

\( [k|M] \sim [k'|M'] \implies \xi_M(k|\cdot) = \xi_{M'}(k'|\cdot) \)
Win when:

\[ k = x_y \]

Average success probability:

\[ S^{rac}(p) = \frac{1}{8} \sum_{x,y} p(x_y | x, y) \]

\[ S^{rac}_{NC} = \frac{3}{4} \]

\[ x = x_0x_1 \in \{0, 1\}^2 \quad y \in \{0, 1\} \]
Win when:

\[ k = x_y \]

Average success probability:

\[
S^{\text{rac}}(p) = \frac{1}{8} \sum_{x,y} p(x_y | x, y)
\]

\[
\frac{1}{2} (P_{00} + P_{11}) \approx \frac{1}{2} (P_{01} + P_{10})
\]

\[
S_{\text{NC}}^{\text{rac}} = \frac{3}{4}
\]
PARITY-OBLIVIOUS
RANDOM ACCESS CODES

Win when:

\[ k = x_y \]

Average success probability:

\[ S^{\text{rac}}(p) = \frac{1}{8} \sum_{x,y} p(x_y | x, y) \]

\[ S^{\text{rac}}_{\text{NC}} = \frac{3}{4} \]

\[ \frac{1}{2}(P_{00} + P_{11}) \approx \frac{1}{2}(P_{01} + P_{10}) \]
Win when:
\[ k = x_y \]

Average success probability:
\[
S^{\text{rac}}(p) = \frac{1}{8} \sum_{x,y} p(x_y | x, y)
\]

\[
S_{\text{NC}}^{\text{rac}} = \frac{3}{4} < S_Q^{\text{rac}} = \frac{1}{2} (1 + \frac{1}{\sqrt{2}})
\]
\[ T = (X, Y, K, \mathcal{OE}_P, \mathcal{OE}_M) \]

\[ p(k | x, y) \quad p \in \mathbb{R}^{XYK} \]

\[ \frac{1}{2}(P_0 + P_1) \simeq \frac{1}{2}(P_2 + P_3) \simeq \frac{1}{2}(P_4 + P_5) \]

\[ \frac{1}{3}([0|M_0] + [0|M_1] + [0|M_2]) \simeq \frac{1}{3}([1|M_0] + [1|M_1] + [1|M_2]) \]
CONTEXTUALITY SCENARIOS THAT “MAP TO” BELL SCENARIOS

Only preparation equivalences

\[ \frac{1}{2} (P_1 + P_2) \simeq \frac{1}{2} (P_3 + P_4) \approx \frac{1}{2} (P_5 + P_6) \checkmark \]

\[ \frac{1}{3} ([0 \mid M_1] + [0 \mid M_2] + [0 \mid M_3]) \simeq \frac{1}{3} ([1 \mid M_1] + [1 \mid M_2] + [1 \mid M_3]) \times \]

AND

Multiple decompositions of one preparation

\[ \frac{1}{2} (P_1 + P_2) \simeq \frac{1}{2} (P_3 + P_4) \approx \frac{1}{2} (P_5 + P_6) \checkmark \]

\[ \frac{1}{2} (P_1 + P_2) \simeq \frac{1}{2} (P_3 + P_4) \text{ AND } \frac{1}{2} (P_5 + P_6) \simeq \frac{1}{2} (P_7 + P_8) \times \]

Liang, Spekkens, Wiseman arXiv:1010.1273
Contextuality inequalities

\( x \in [X] \)

\( y \in [Y] \)

\( k \in [K] \)

\( p(a, b | x, y) \)

\( p(k | x, y) \)

\( S(p) \leq S_{NC} \)

Contextuality inequalities
There exist hierarchies of semidefinite programs to approximate $\mathcal{Q}$ from the outside (like NPA for Bell scenarios).

But not much else is known about $\mathcal{Q}$.


Navascues, Pironio, Acín arXiv:0803.4290
No-signalling B → A

\[ p(11|00) + p(10|00) = p(10|01) + p(11|01) \]
No-signalling $B \rightarrow A$

\[
p(11|00) + p(10|00) = p(10|01) + p(11|01) = 1 - p(00|01) - p(01|01)
\]
No-signalling $B \rightarrow A$

\[
p(11|00) + p(10|00) \\
= p(10|01) + p(11|01) \\
= 1 - p(00|01) - p(01|01)
\]
No-signalling $B \rightarrow A$

\[ p(11|00) + p(10|00) = p(10|01) + p(11|01) = 1 - p(00|01) - p(01|01) \]
\[ p(11|00) + p(10|00) \]
\[ = p(10|01) + p(11|01) \]
\[ = 1 - p(00|01) - p(01|01) \]
Quantum strategy

<table>
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$|0\rangle$ $|f\rangle$ $|\bar{0}\rangle$ $|e\rangle$ $|\bar{f}\rangle$ $|1\rangle$ $|E\rangle$ $|F\rangle$

$|\bar{1}\rangle$ $|\bar{e}\rangle$ $|\bar{f}\rangle$ $|\bar{E}\rangle$ $|\bar{F}\rangle$ $|\bar{1}\rangle$ $|\bar{E}\rangle$ $|\bar{F}\rangle$

CHSH
Quantum strategy

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</table>

$|f\rangle$ $|e\rangle$ $|0\rangle$ $|1\rangle$

$|E\rangle$ $|e\rangle$ $|0\rangle$ $|1\rangle$

$|F\rangle$ $|e\rangle$ $|0\rangle$ $|1\rangle$

$|0\rangle$ $|0\rangle$ $|0\rangle$ $|0\rangle$

$|1\rangle$ $|1\rangle$ $|1\rangle$ $|1\rangle$

$|01|00\rangle$ $|10|00\rangle$ $|00|01\rangle$ $|01|01\rangle$

$|11|00\rangle$ $|10|01\rangle$ $|10|01\rangle$ $|11|01\rangle$

$|00|10\rangle$ $|01|10\rangle$ $|11|11\rangle$ $|10|11\rangle$

$|00|11\rangle$ $|11|10\rangle$ $|10|11\rangle$ $|00|11\rangle$
Quantum strategy

Local correlations $\cong$ NC statistics