COPENHAGEN 2022 CONNES

VERN I. PAULSEN

Abstract. These notes are intended for those students that would like to see the quickest route for the proof that the Connes Embedding Problem has a negative answer.

1. About the Connes Embedding Problem, Von Neumann Algebras, and All That

The Connes Embedding Problem(CEP) was an open question about the structure of a certain kind of von Neumann algebra. Due to the work of Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, Henry Yuen in the paper MIP*= RE[5], we now know that this problem has a negative answer. The purpose of this first lecture is to introduce us to what this problem was all about. Attempts to resolve this problem lead to the development of many equivalent restatements of the problem, which now also all have negative answers. In fact, the paper MIP*=RE does not negate the original problem, but one of the equivalent reformulations. Sorting through all these equivalent reformulations is a long road, and to negate the original statement of the problem we do not need the equivalences but just implications in one direction. Often the only direction that we need is actually the "easier" of the two implications. So our other goal for today is to sketch in the shortest route from the CEP to the version that we will negate, all while keeping the discussion at a level accessible to non-experts in operator algebras. As a bonus, we will obtain an even stronger refutation of the CEP than was first realized. What we now call Von Neumann Algebras were introduced by von Neumann which he called Algebras of Operators, but their name has since been changed in his honor. His motivation for studying these was quantum mechanics, so we start with that viewpoint. Quantum mechanical systems come with a Hilbert space called the state space and the unit vectors in that Hilbert space represent the pure states of a system. Suppose that the system is in state $\psi \in \mathcal{H}$ and we want to perform a measurement that has K possible outcomes. Then the standard model says that there will exist K measurement operators, $M_1, ...M_K \in B(H)$ such that:

- the probability of getting outcome k is $p_k = \|M_k\psi\|^2$,
- if outcome k is observed then the state of the system changes to $\frac{M_k\psi}{\|M_k\psi\|}$

The fact that $1 = \sum_{k=1}^K p_k$ implies that

$$1 = \sum_{k=1}^K \langle M_k\psi|M_k\psi\rangle = \sum_{k=1}^K \langle \psi|M_k^* M_k\psi\rangle$$

which implies that

$$\sum_{k=1}^K M_k^* M_k = I.$$

Von Neumann argued that in certain settings the underlying state space could have a family of unitaries that acted upon it $\{U_a : a \in A\}$. Think perhaps of the case where these unitaries represent moving the state to a different location in space. Suppose you know that if you first measured the state and then moved it to a new location, then this should be the same as if you first moved it to the new location and then measured the state. Intuitively, the outcome of the quantum experiment is not changed by where in your lab you perform it. If this is the case then any measurement operator M for this system must satisfy, $MU_a = U_a M, \forall a$. Thus, the measurement operators for this system should not be all operators on the state space, but a subset of $B(H)$.

This lead him to study sets of operators that commute with a set of unitary operators. He denoted such sets by $\mathcal{M}$ since this is what he thought sets of measurement operators should look like. For this reason von Neumann algebras are still generally denoted by the letter $\mathcal{M}$.

Definition 1.1. Given a set $\mathcal{S} \subseteq B(\mathcal{H})$ we call the set

$$\mathcal{S}' = \{T \in B(\mathcal{H}) : TS = ST, \forall S \in \mathcal{S}\},$$

the **commutant** of $\mathcal{S}$.

Problem 1.2. *Show that*

$$\mathcal{S}' = \mathcal{S}''',$$

and

$$\mathcal{S}'' = \mathcal{S}''''.$$

We briefly recall weak, strong and weak* convergence. A net of operators $\{T_\lambda\}_{\lambda\in D}$ converges to T in

- the weak-topology if $\langle k|T_\lambda h\rangle \to \langle k|Th\rangle$ for all $h, k \in \mathcal{H}$,
- the strong-topology if $\|T_\lambda h - Th\| \to 0$ for all $h \in \mathcal{H}$,
- the weak*-topology if $Tr(T_\lambda K) \to Tr(TK)$ for all $K \in \mathcal{C}_1(\mathcal{H})$. Here $\mathcal{C}_1(\mathcal{H})$ denotes the trace class operators.

We use $\mathcal{S}^{-w}$, $\mathcal{S}^{-s}$ and $\mathcal{S}^{-wk*}$, to denote the sets of operators that are limits of nets of operators from $\mathcal{S}$ in the weak, strong, and weak* sense.

Recall that a set $\mathcal{A}$ is called an *algebra* if it is a vector space and $X, Y \in \mathcal{A} \implies XY \in \mathcal{A}$.

Problem 1.3. *Let $\mathcal{S} \subseteq B(\mathcal{H})$ be a set. Then $\mathcal{S}'$ is a subalgebra of $B(\mathcal{H})$. For those with a functional analysis background, show that $\mathcal{S}'$ is closed in the weak, strong and weak* topologies.*

Problem 1.4. *Show that if $\mathcal{S} \subseteq B(\mathcal{H})$ is a set of unitaries and $T \in \mathcal{S}'$ then $T^* \in \mathcal{S}'$. Such a set is said to be* ***-closed**.

These two problems show that commutants of sets of unitaries are subalgebras that are *-closed and closed in many topologies.

Theorem 1.5 (von Neumanns bicommutant Theorem)**.** *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an algebra of operators such that $I \in \mathcal{A}$ and $X \in \mathcal{A} \implies X^* \in \mathcal{A}$. Then $\mathcal{A}'' = \mathcal{A}^{-w} = \mathcal{A}^{-s} = \mathcal{A}^{-wk*}$.*

Thus, not only are operators that can be realized as limits in these three senses all equal, but something defined purely algebraically, $\mathcal{A}''$ is equal to something defined as a topological closure.

Corollary 1.6. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be an algebra of operators such that $I \in \mathcal{M}$ and $X \in \mathcal{M} \implies X^* \in \mathcal{M}$. The following are equivalent:*

- $\mathcal{M} = \mathcal{M}''$,
- $\mathcal{M} = \mathcal{M}^{-s}$,
- $\mathcal{M} = \mathcal{M}^{-w}$,
- $\mathcal{M} = \mathcal{M}^{-wk*}$.

Definition 1.7. Any $\mathcal{M} \subseteq B(\mathcal{H})$ satisfying $I \in \mathcal{M}$, $X \in \mathcal{M} \implies X^* \in \mathcal{M}$ and $\mathcal{M} = \mathcal{M}''$ is called a **von Neumann algebra**.

Murray and von Neumann set about to classify all such algebras. This program goes on to this day and has an influence on quantum mechanics, which we will try to explain. First, we discuss the classification program.

Definition 1.8. A von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is called a **factor** if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C} \cdot I$.

Just like all integers can be decomposed into products of primes, Von Neumann proved that every von Neumann algebra could be built from factors by something called **direct integration theory**. Hence, to understand all von Neumann algebras we only need to understand all factors.

The next step that Murray and von Neumann made was to break factors down into three types.

Definition 1.9. Let $\mathcal{M}$ be a von Neumann algebra and let $E, F \in \mathcal{M}$ be projections. We write $E \le F$ if $\mathcal{R}(E) \subseteq \mathcal{R}(F)$ and $E < F$ when $E \le F$ and $E \neq F$. We say that $F \neq 0$ is **minimal** if $E < F \implies E = 0$. We say that E and F are **Murray-von Neumann equivalent** and write $E \sim F$ if there exists a partial isometry $W \in \mathcal{M}$ such that $E = W^*W, F = WW^*$. We say that F is **finite** if there is no E such that $E \sim F$ and $E < F$.

Some examples are useful.

Example 1.10. Let $\mathcal{M} = B(\mathbb{C}^m)$. Then F is minimal if and only if F is a rank one projection, $E \sim F$ if and only if E and F are projections onto subspaces of the same dimension. Hence, every projection is finite.

Example 1.11. Let $\mathcal{M} = B(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. Again a projection is minimal if and only if it is rank one. Since a partial isometry with $E = W^*W, F = WW^*$ is an isometry from $\mathcal{R}(E)$ to $\mathcal{R}(F)$ and isometries preserve inner products, they take an onb for one space to an onb to the other space. Hence, $E \sim F \iff dim_{HS}(\mathcal{R}(E)) = dim_{HS}(\mathcal{R}(F))$. On the other hand as soon as a set is infinite we can throw away one element and the subset has the same cardinality. Hence, as soon as $\mathcal{R}(F)$ is infinite dimensional, we can take an onb, throw away one element and that will be an onb for a subspace $\mathcal{R}(E) \subseteq \mathcal{R}(F)$ of the same dimension. We can now take a partial isometry that sends the onb for $\mathcal{R}(F)$ to the onb for $\mathcal{R}(E)$. This shows that $E \sim F$ with $E < F$. Hence, F is NOT finite in the M-vN sense.

Hence, the only finite projections are the projections onto finite dimensional subspaces.

Definition 1.12. A von Neumann factor is called **Type I** if it has a minimal projection. It is **Type II** if it has no minimal projections, but has a finite projection. It is called **Type** II_1 is it is Type II and the identity is a finite projection. If it is Type II but not Type Type II_1, then it is called **Type** II_∞. It is called **Type III** if it is not Type I or Type II.

Theorem 1.13 (Murray-von Neumann)**.** *A von Neumann algebra is Type I if and only if it is *-isomorphic to $B(\mathcal{H})$ for some $\mathcal{H}$.*

Definition 1.14. Let $\mathcal{M}$ be a von Neumann algebra. A map $\tau : \mathcal{M} \to \mathbb{C}$ is called a **state** is $\tau(I) = 1$ and $p \geq 0 \implies \tau(p) \geq 0$. A map is called a **tracial state** if it is a state and satisfies $\tau(XY) = \tau(YX)$. It is **faithful** if $\tau(X^*X) = 0 \implies X = 0$.

For $B(\mathbb{C}^n)$, there is only one faithful tracial state, namely $\tau_n(X) = \frac{1}{n}Tr(X)$. Note that in this setting the possible traces of projections are the numbers $\{\frac{k}{n} : 0 \leq k \leq n\}$, which represent the "fractional" dimension of the corresponding subspace.

Theorem 1.15 (Murray-von Neumann)**.** *Let $\mathcal{M}$ be a Type II_1 factor. Then:*

- *there exists a faithful tracial state, $\tau : \mathcal{M} \to \mathbb{C}$ that is also continuous in the weak*-topology,*
- *for every $0 \leq t \leq 1$ there exists a projection $P \in \mathcal{M}$ with $\tau(P) = t$,*
- *if $P, Q \in \mathcal{M}$ are projections, then $P \sim Q \iff \tau(P) = \tau(Q)$.*

This lead von Neumann to refer to *continuous geometries* since, unlike finite dimensions, in a Type II_1 setting there are subspaces of every (fractional) dimension t for every $0 \leq t \leq 1$. Thus, even projections of irrational dimension exist.

Many questions in this classification program are now known to hinge on the classification of Type II_1-factors. One of the great breakthroughs in this theory was made by Alain Connes, who studied II_1-factors that , in a certain sense, are limits of matrix algebras. Such II_1-factors are called *hyperfinite*.

Connes proved that all hyperfinite II_1-factors are isomorphic in an appropriate sense. This one common hyperfinnite II_1-factor is denoted by $\mathcal{R}$.

We describe one way to obtain this algebra. Suppose that our Hilbert space is $\mathcal{H} = \ell^2(\mathbb{N})$ with orthonormal basis $\{e_n = |n\rangle : n \in \mathbb{N}\}$. By dividing the integers into the even and odd integers we obtain two subspaces $\mathcal{H}_{even}, \mathcal{H}_{odd}$ such that $\mathcal{H} = \mathcal{H}_{even} \oplus \mathcal{H}_{odd}$. Moreover, the map $e_n \to e_{n+1}$ defines an isomorphism $W : \mathcal{H}_{odd} \to \mathcal{H}_{even}$.

If we map a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the operator $aP_{odd}+bW+cW^*+dP_{even}$ then we see that this is a *-isomorphism of M_2 into $B(\mathcal{H})$. Thus, modulo this identification, we have $M_2 \subseteq B(\mathcal{H})$.

Now repeat this process by dividing the evens and odds in half. That is, consider the four sets that are the integers modulo 4. In this way we get a copy of the 4×4 matrices included into $B(\mathcal{H})$ with

$$M_2 \subseteq M_4 \subseteq B(\mathcal{H}).$$

Note that the way that the inclusion of M_2 into M_4 happens is that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Note that when we take the unique tracial states on M_2 and M_4 then this map preserves that tracial state since

$$\tau_2(\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2} = \tau_4(\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}).$$

Continuing this process, we obtain copies of $M_{2^n} \subseteq B(\mathcal{H})$ for all n in such a way that $M_{2^n} \subseteq M_{2^{n+1}}$ via a similar doubling map that preserves these normalized traces.

The union of all the operators obtained this way defines a subset of $B(\mathcal{H})$, often denoted by M_{2^∞}. It is not hard to see that this set is an algebra and that if X is in this set, then X^* is in this set. Hence, M_{2^∞}'' is a von Neumann algebra, that is in a certain sense a limit of matrix algebras.

Since the inclusions preserve traces at each stage, there is a well-defined map $\tau_{2^\infty} : M_{2^\infty} \to]\mathbb{C}$ given by

$$\tau_{2^\infty}(X) = \tau_{2^n}(X) \text{ for } X \in M_{2^n}.,$$

This map extends to define a canonical tracial state $\tau : M''_{2^\infty} \to \mathbb{C}$.

In fact, M''_{2^∞} is a hyperfinite II_1-factor. So this is one very concrete way to obtain the algebra $\mathcal{R}$ along with its trace τ.

Now if instead of dividing the integers into "halves", we had used the integers modolo $3, 3^2, ...$, we would obtain a union of subalgebras denoted $M_{3^\infty} \subseteq B(\mathcal{H})$ and a new hyperfinite von Neumann algebra M''_{3^∞} along with a canonical trace and these two algebras would be isomorphic in a trace preserving fashion by Connes' theorem. So $\mathcal{R}$ could be obtained this way as well.

In fact, these two von Neumann algebras were shown to isomorphic by von Neumann. Moreover, it is known that if we do not take the double commutant, then M_{2^∞} and M_{3^∞} are not isomorphic. In fact, even if we just take their norm closures in $B(\mathcal{H})$, then these are two C*-algebras that are not isomorphic.

After successfully describing all hyperfinite II_1-factors, Connes speculated on how one might obtain all Type II_1-factors. Given any von Neumann algebra with trace $(\mathcal{M}, \tau)$ there is a process by which one can build a larger algebra and trace, called taking the *ultrapower* of the algebra, denoted $(\mathcal{M}^\omega, \tau_\omega)$. This construction is a bit too complicated to get into here, but we will quickly pass to some equivalences.

Connes' Embedding Problem(CEP) asks if given an arbitrary Type II_1-factor and trace $(\mathcal{M}, \tau)$, does it embed in a trace preserving manner into $(\mathcal{R}^\omega, \tau_\omega)$?

Here embeds means that we seek a map $\pi : \mathcal{M} \to \mathcal{R}^\omega$ that is a unital, *-homomorphism, that is continuous with respect to the weak*-topologies, and is *trace preserving*, i.e.,

$$\tau_\omega(\pi(X)) = \tau(X), \ \ \forall X \in \mathcal{M}.$$

Later Haagerup showed that if CEP was true then for every separable C*-algebra with a trace $(\mathcal{A}, \tau)$ there would exist a *-homomorphism into $(\mathcal{R}^\omega, \tau_\omega)$ that is trace preserving manner. This fact allows us to not have to think about weak*-topologies. This is one of the many equivalent restatements of the CEP. We take this as our formal statement of the CEP.

Connes Embedding Problem: Given a separable C*-algebra and a tracial state $(\mathcal{A}, \tau)$ does there exist a *-homomorphism $\pi : \mathcal{A} \to \mathcal{R}^\omega$ such that $\tau = \tau_\omega \circ \pi$?

The *matricial microstates problem* is another restatement that avoids all discussion of ultrapowers. Informally, it asks if traces of products of self-adjoint elements up to a certain length can be approximated by traces of self-adjoint matrices of a large enoguh size. From now on we use $\tau_n : M_n \to \mathbb{C}$ to denote the unique normalized trace, $\tau_n(X) = 1/n \sum_{i=1}^n x_{i,i}$.

The formal statement is below.

1.1. **Matricial Microstates.** . Given a C*-algebra and tracial state $(\mathcal{A}, \tau)$, n self-adjoint elements $h_1, ..., h_n \in \mathcal{A}$ of norm 1, an integer k and $\epsilon > 0$, a

(k,ϵ)-*matricial microstate* is a matrix algebra M_d and self-adjoint matrices $H_1, ..., H_n \in M_d$ of norm 1, such that

$$|\tau(h_{i_1} \cdots h_{i_j}) - \tau_d(H_{i_1} \cdots H_{i_j})| < \epsilon,$$

holds for every possible product of $j \leq k$ elements.

When this is true for $(\mathcal{A},\tau)$ for all n, k, ϵ then we say that $(\mathcal{A},\tau)$ *has matricial microstates.*. Here is the key fact.

Theorem 1.16 (Voiculsecu [13]). *Let $(\mathcal{A},\tau)$ be a C*-algebra and tracial state. Then there exists a trace preserving *-homomorphism of $(\mathcal{A},\tau)$ into $(\mathcal{R}^\omega, \tau_\omega)$ if and only if $(\mathcal{A},\tau)$ has (k,ϵ)-matricial microstates for all k, ϵ and for all sets of n self-adjoint elements of norm 1.*

Thus, loosely speaking, the CEP is equivalent to a problem about approximating traces of words in self-adjoint elements in a generic tracial C*-algebra $(\mathcal{A},\tau)$ by traces of words in Hermitian matrices.

The proof of the above theorem uses the fact that there are trace preserving embeddings of matrix algebras into $(\mathcal{R}^\omega, \tau_\omega)$ in such a way that every finite set of elements in $\mathcal{R}^\omega$ can be approximated in a good sense by matrices in the image.

Here is the consequence of CEP that the paper MIP*=RE contradicts.

By an (n,k) *projection valued measure((n,k)-PVM) in* $\mathcal{A}$ we mean a set of nk projections $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\} \subseteq: \mathcal{A}$ with $\sum_{a=1}^k e_{x,a} = I_\mathcal{A}$ for every x.

Theorem 1.17 (Dykema-P[1]). *If CEP has an affirmative answer, then for every $(\mathcal{A},\tau)$, every (n,k)-PVM $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\} \subseteq \mathcal{A}$ and every $\epsilon > 0$, there is a matrix algebra M_d and an (n,k)-PVM $\{E_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\} \subseteq M_d$ such that*

$$|\tau(e_{x,a}e_{y,b}) - \tau_d(E_{x,a}E_{y,b})| < \epsilon.$$

The converse of this result is also true, but it uses other deeper results, see [1, Theorem 3.7].

We sketch the key ideas of the proof. The proof that we present here is more direct than given in [1], but a bit longer.

If CEP holds, then there are matricial microstates. So there is some matrix algebra M_d and approximating self-adjoint contraction matrices $\{H_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\} \subseteq M_d$. We need to somehow replace these generic self-adjoint matrices by a (n,k)-POVM.

First note that since $e_{x,a}^2 = e_{x,a}$, every word in the $e_{x,a}$'s can be replaced by a word in $e_{x,a}^2$ which is approximated by the corresponding word in $P_{x,a} = H_{x,a}^2$. So, by doubling the word length, without loss of generality, we may assume that we have positive contractions $\{P_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ for our microstate.

Using words of length 2 in these new generators, we may assume that $tr_d(P_{x,a}) \approx \tau(e_{x,a})$ and $tr_d(P_{x,a}^2) \approx \tau(e_{x,a}^2) = \tau(e_{x,a})$. Thus, $tr_d(P_{x,a} -$

$P_{x,a}^2) \approx 0$ and this is enough to imply that "most" of the eigenvalues of $P_{x,a}$ are clustered around 0 and 1.

So we replace $P_{x,a}$ by the projections $E_{x,a}$ onto the subspace of eigenvalues "near" to 1.

Next we show that for $a \neq b$,

$$tr_d(E_{x,a}E_{x,b}) \approx tr_d(P_{x,a}P_{x,b}) \approx \tau(e_{x,a}e_{x,b}) = 0,$$

and this allows us to replace the $E_{x,a}$ by nearby projections satisfying $E_{x,a}E_{x,b} = 0$.

Finally, since $tr_d(\sum_a E_{x,a}) \approx \tau(\sum_a e_{x,a}) = 1$ a further approximation allows us to assume that $\sum_a E_{x,a} = I_d$.

Problem 1.18. *Let $P \in M_d$ be a positive contraction and assume that $tr_d(P - P^2) < \epsilon$. Prove that if we let E be the projection onto the eigenspace of all eigenvalues of P that are greater than $1 - \sqrt{\epsilon}$, then*

$$tr_d(|E - P|) < 2\sqrt{\epsilon}.$$

(Hint: First calculate how many eigenvalues of P can lie in the interval $[\sqrt{\epsilon}, 1 - \sqrt{\epsilon}]$.)

Proof. Let $1 \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$ denote the eigenvalues of P. Note that if $\lambda \in [\sqrt{\epsilon}, 1 - \sqrt{\epsilon}]$, then $\lambda - \lambda^2 \geq \sqrt{\epsilon} - \epsilon$. If P has r eigenvalues in this range then

$$r(\sqrt{\epsilon} - \epsilon) \leq \sum_{i=1} \lambda_i - \lambda_i^2 = Tr(P - P^2) < d\epsilon.$$

Hence,

$$r < \frac{d\epsilon}{\sqrt{\epsilon} - \epsilon} = \frac{d\sqrt{\epsilon}}{1 - \sqrt{\epsilon}}.$$

Say that $1 \geq \lambda_1 \geq \cdots \lambda_k \geq 1 - \sqrt{\epsilon}$, so that E is the projection onto the eigenspace of these eigenvalues. Then

$$Tr(|E - P|) = \sum_{\lambda_i \geq 1 - \sqrt{\epsilon}} (1 - \lambda_i) + \sum_{\sqrt{\epsilon} \leq \lambda_i \leq 1 - \sqrt{\epsilon}} \lambda_i + \sum_{\lambda_i \leq \sqrt{\epsilon}} \lambda_i \leq$$
$$k\sqrt{\epsilon} + r(1 - \sqrt{\epsilon}) + (d - k - r)\sqrt{\epsilon} < (d - r)\sqrt{\epsilon} + d\sqrt{\epsilon},$$

and the result follows. □

2. The Problems of Tsirelson and Kirchberg

Kirchberg proved that CEP had an affirmative answer if and only if there was a unique norm on the tensor products of certain group C*-algebras. We won't spend much time on this equivalence here, since it is not essential to our presentation. But it was very important to further developments. Scholz and Werner were the first to realize that Kirchberg's problem was related to a question raised by Tsirelson about the mathematical description of quantum correlations.

The statement below is slightly more general than can be found in [8] but is an easy extension.

Theorem 2.1 (Kirchberg)**.** *Let $\mathcal{C}$ be the set of groups that are free products of cyclic groups, other than $\mathbb{Z}_2 \star \mathbb{Z}_2$. Then the following are equivalent:*

(1) *CEP has an affirmative answer,*
(2) *for every pair of groups, $G, H \in \mathcal{C}$, there is a unique C^*-norm on $C^*(G) \otimes C^*(H)$,*
(3) *there exists a pair of groups $G, H \in \mathcal{C}$, such that there is a unique C^*-norm on $C^*(G) \otimes C^*(H)$.*

When we refer to Kirchberg's problem(s) we are referring to (2) or (3). We now turn our attention to Tsirelson's problem.

Suppose that Ali

Suppose that Alice and Bob have separated, isolated labs and they can each perform one of n_A, respectively, n_B, quantum measurements and each measurement has, respectively, k_A and k_B outcomes. We let $p(a,b|x,y)$ denote the conditional probability density that Alice gets outcome a and Bob gets outcome b, when the perform measurements x and y, respectively. Such densities are also called **quantum correlations** and Tsirelson was interested in mathematical descriptions of the set of all such conditional densities.

It turns out that the axiomatic quantum theory allows for several possible mathematical descriptions of these sets of densities and Tsirelson was interested in whether these were all the same. So we start with the possible descriptions.

The basic quantum model assumes that Alice and Bob labs are described by finite dimensional state spaces, $\mathcal{H}_A, \mathcal{H}_B$ and that the state of their combined labs is given by a unit vector $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Alice's and Bob's measurements are each given by an (n,k)-POVM, $\{E_{x,a} : 1 \le x \le n_A, 1 \le a \le k_A\}$ and $\{F_{y,b} : 1 \le y \le n_B, 1 \le b \le k_B\}$, which means we have families of projections such that

$$\sum_{a=1}^{k_A} E_{x,a} = I_{\mathcal{H}_A}, \forall x, \text{ and } \sum_{b=1}^{k_B} F_{y,b} = I_{\mathcal{H}_B}, \forall y,$$

and

$$p(a,b|x,y) = \langle \psi | (E_{x,a} \otimes F_{y,b}) \psi \rangle.$$

We let $C_q(n_A, n_B, k_A, k_B)$ denote the set of all $p(a,b|x,y)$ that can be obtained as above, which we call the **quantum correlations** or **quantum densities**. Note that since $0 \le p(a,b|x,y) \le 1$ that we can always regard $C_q(n_A, n_B, k_A, k_B)$ as a subset of the compact set $[0,1]^{n_A n_B k_A k_B}$. Generally, we shall be interested in the case that $n_A = n_B = n$ and $k_A = k_B = k$, in which case we shorten this to $C_q(n,k)$.

A slightly more general model is to allow $\natural_A$ and $\mathcal{H}_B$ to be arbitrary Hilbert spaces in which case we denote this larger set by $C_{qs}(n_A, n_B, k_A, k_B)$ where the subscript stands for **quantum spatial**.

There is no reason that either of these sets needs to be closed. However, a nice result that uses C*-algebra theory is that they both have the same closure and we set

$$C_{qa}(n_A, n_B, k_A, k_B) := C_q(n_A, n_B, k_A, k_B)^- = C_{qs}(n_Q, n_B, k_A, k_B).$$

These are called the **quantum approximate** correlations.

An even more general model is to assume that the combined state space of Alice and Bob does not decompose as a tensor product but instead that it is a single Hilbert space $\mathcal{H}$ so that they each have POVM's on this space,

$$\{E_{x,a} : 1 \le x \le n_A, 1 \le a \le k_A\} \subseteq B(\mathcal{H}),\ \{F_{y,b} : 1 \le y \le n_B, 1 \le b \le k_B\} \subseteq B(\mathcal{H}),$$

with the property that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a},\ \forall x, y, a, b$. We call this a commuting model. Note that it is only Alice's operators that must commute with Bob's operators. There is no requirement that Alice's operators commute among themselves for different inputs.

The set of all

$$p(a, b|x, y) = \langle \phi | E_{x,q} F_{y,b} \phi \rangle,$$

that can be obtained in this manner for some commuting model and some unit vector ϕ is denoted $C_{qc}(n_A, n_B, k_A, k_B)$ and called the **quantum commuting** correlations.

The explanation for this commuting hypothesis is that the outcome should not depend on the order of applying their measurements. Note that if in the tensor cases we have that

$$E_{x,a} \otimes F_{y,b} = (E_{x,a} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes F_{y,b}) = (I_{\mathcal{H}_A} \otimes F_{y,b})(E_{x,a} \otimes I_{\mathcal{H}_B}).$$

so it is a commuting correlation. Thus, we have

$$C_q(n_A, n_B, k_A, k_B) \subseteq C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{qa}(n_A, n_B, k_A, k_B) \subseteq C_{qc}(n_A, n_B, k_A, k_B).$$

In the case that $n_A = n_B = k_A = k_B = 2$, Tsirelson proved that these sets are all equal, and wondered if this could be true more generally.

The first great insight was [**?**] who noted the connection between Kirchberg's problems and Tsirelson's problems in a short note. This was then developed fully in [6] and [9].

Theorem 2.2 (Junge, Navascues, Palazuelas, Perez-Garcia, Scholz, Werner[6]; Ozawa[9])**.** *Kirchberg's problems(and consequently CEP) have an affirmative answer if and only if*

$$C_{qa}(n, k) = C_{qc}(n, k),\ \forall n, k.$$

Thuis, one of the problems introduced by Tsirelson, is equivalent to the CEP. This ignited a lot of interest in the study of the relationship between these various models for correlation.

The first breakthrough was due to Slofstra.

Theorem 2.3 (Slofstra [11])**.** *For n, k large enough, the set $C_q(n, k)$ is not closed.*

The values of n, k for Slofstra's proof to apply seem to be $n \sim 200$, $k \sim 8$. Shortly after Slofstra's result it was found that this non-closure is true even for small values and how pathologically non-closed these sets are was revealed.

Theorem 2.4 (Dykema-P-Prakash[2])**.** *The sets $C_q(n,k)$ are not closed for every $n \geq 5, k \geq 2$. Let $\frac{\sqrt{5}-1}{2\sqrt{5}} \leq t \leq \frac{\sqrt{5}+1}{2\sqrt{5}}$ and for $0 \leq a,b \leq 1,\ 1 \leq x,y \leq 5$ set*

$$p(0,0|x,x) = t, \qquad p(0,1|x,x) = p(1,0|x,x) = 0, \qquad p(1,1|x,x) = 1-t,$$

and for $x \neq y$, set

$$p(0,0|x,y) = \frac{1}{4}t(5t-1), \qquad p(0,1|x,y) = p(1,0|x,y) = \frac{5}{4}t(1-t),$$
$$p(1,1|x,y) = \frac{1}{4}(1-t)(4-5t).$$

Then $p \in C_{qa}(5,2)$ for all t in this interval, but $p \in C_q(5,2)$ only for t is rational.

Note that this is a nice continuous path of correlations p_t but to "decide" if p_t belongs to $C_q(5,2)$ one must be able to decide if t is rational. For example it is still unknown if $e+\pi$ is rational. So we can write down formal expressions for t for which it is still unknown if p_t belongs to $C_q(5,2)$.

By Tsirelson's results, $C_q(2,2)$ is closed, but it is still not known if $C_q(3,2)$ and $C_q(4,2)$ are closed.

The paper MIP*=RE proved that there exists n, k such that $C_{qa}(n,k) \neq C_{qc}(n,k)$ and hence by [6] CEP has a negative answer. For the purposes of proving that CEP is negative one only needs the implications:

$$CEP \text{ affirmative } \implies KP \text{ affirmative } \implies C_{qa}(n,k) = C_{qc}(n,k),\ \forall n,k$$

and the second implication is not too difficult for students with sufficient background. However, the first implication, Kirchberg's result, is quite hard. The relevant free group in this setting is the free product of n copies of the cyclic group of order k, denoted $\mathbb{F}(n,k)$.

The following problems indicate the connection between free groups and these correlations.

Problem 2.5. *Let $\omega = e^{2\pi i/k}$. Given projections $E_a, 1 \leq a \leq k$ with $\sum_{a=1}^k E_a = I$, show that*

$$U = \sum_{a=1}^k \omega^a E_a,$$

is a unitary of order k. Conversely, if $U \in B(\mathcal{H})$ is a unitary with $U^k = I$ and we set

$$E_a = \sum_{b=1}^k (\omega^{-a} U)^b,$$

then these are orthogonal projections summing to I. Use this to show that there is a correspondence between (n,k)-PVM's and representations of $\mathbb{F}(n,k)$.

Problem 2.6. *(This is for students with a background in C*-algebras) Show that the sets $C_{qc}(n,k)$ and $C_q(n,k)$ correspond to two families of representations of $\mathbb{F}(n,k)\times\mathbb{F}(n,k)$, that these two families of representations yield two possibly different seminorms on $C^*(\mathbb{F}(n,k))\otimes C^*(\mathbb{F}(n,k))$. Finally, deduce that if these two seminorms are equal, then $C_{qa}(n,k) = C_{qc}(n,k)$.*

To complete the proof that if KP is affirmative, then these correlation sets are equal, one needs to know that these seminorms are actually norms. The fact that these are indeed norms follows from the fact that the group $\mathbb{F}(n,k)$ is residually finite dimensional(RFD).

3. Synchronous Correlations and a Simpler Refutation of CEP

In these lectures we will show a proof that CEP is negative that avoids the use of Kirchberg's equivalences and instead uses 1.17. We will achieve this by using a connection between traces and correlations discovered in [10].

A correlation $p \in C_{qc}(n,k)$ is called **synchronous** provided that

$$p(a,b|x,x) = 0, \; \forall a \neq b.$$

Thus, whenever Alice and Bob simultaneously perform the same experiment, they must get the same outcome. We use a superscript s for the subset of synchronous correlations.

Here is the key theorem connecting traces and synchronous densities.

Theorem 3.1 (P-Severini-Stahlke-Todorov-Winter [10])**.** (1) *$p \in C^s_{qc}(n,k)$ if and only if there exists a tracial C*-algebra $(\mathcal{A},\tau)$ and an (n,k)-PVM $\{e_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ in $\mathcal{A}$ such that*

$$p(a,b|x,y) = \tau(e_{x,a}e_{y,b}).$$

(2) *$p \in C_q(n,k)$ if and only if in the above representation we can assume that $\mathcal{A}$ is finite dimensional.*

We sketch one of the key ideas of the proof. Suppose that we have written

$$p(a,b|x,y) = \langle \phi | E_{x,a}F_{y,b}\phi\rangle,$$

then

$$1 = \sum_{a,b=1}^k p(a,b|x,x) = \sum_{a=1}^k p(a,a|x,x) = \sum_{a=1}^k \langle E_{x,a}\phi|F_{x,a}\phi\rangle \le$$
$$\sum_{a=1}^k \|E_{x,a}\phi\|\cdot\|F_{x,a}\phi\| \le (\sum_{a=1}^k \|E_{x,a}\phi\|^2)^{1/2}(\sum_{a=1}^k \|F_{x,a}\phi\|^2)^{1/2} = 1.$$

Thus, the inequality is an equality and this in turn implies that $E_{x,a}\phi = F_{x,a}\phi$, $\forall x, a$.

Using this one shows that if we let $\mathcal{A}$ be the C*-algebra generated by $\{E_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ and let $\tau : \mathcal{A} \to \mathbb{C}$ be the state given by $\tau(X) = \langle \phi | X\phi \rangle$, then $\tau(XY) = \tau(YX)$, i.e., τ is a trace.

This proves one direction of (1). The converse, that setting $p(a,b|x,y) = \tau(E_{x,a}E_{y,b})$ when τ is a trace defines an element of $C_{qc}(n,k)$ is a standard argument for experts in C*-algebras.

Note that if $p \in C_q^s(n,k)$ then the $E_{x,a}$ would all be matrices and so $\mathcal{A}$ would be a finite dimensional C*-subalgebra of this matrix algebra. However, this does not imply that τ is the usual trace, unlike full matrix algebras where the trace is unique, subalgebras can have many traces.

However, this last result allows us to restate 1.17 as follows:

Theorem 3.2 (Dykema-P)**.** *If CEP has an affirmative answer, then for every n,k the closure of $C_q^s(n,k)$ is equal to $C_{qc}^s(n,k)$.*

To see this note that by 1.17, every $p \in C_{qc}^s(n,k)$ can be approximated by traces on the full matrix algebra M_d of some (n,k)-PVM's in M_d for d sufficiently large.

The results of MIP*=RE, using the theory of games, imply that for some n,k there is an element of $C_{qc}^s(n,k)$ that is not in the closure of $C_q(n,k)$. Thus, taking this fact as a given, we have given a fairly complete sketch of why their results imply that the matricial microstates version of CEP is false.

Our next goal is to expand on the topic of games and, especially, perfect strategies for games and values of games. But before doing that we state, without sketching proofs, the deeper results about synchronous correlations that give rise to the stronger refutation of CEP.

Note that the closure of $C_q^s(n,k)$, denoted $(C_q^s(n,k))^-$ is a subset of $C_{qa}^s(n,k)$, since the second set is the set of correlations that are limits of sequences of (not necessarily synchronous) correlations, whose limit is synchronous. The results of [10] also leads one to wonder about characterizations of the other synchronous correlation sets. These questions were all answered in [7].

Theorem 3.3 (Kim-P-Schafhauser)**.** *We have the following:*

(1) $(C_q^s(n,k))^- = C_{qa}^s(n,k)$, $\forall n,k$,
(2) $C_q^s(n,k) = C_{qs}^s(n,k)$, $\forall n,k$,
(3) $p \in .C_{qa}^s(n,k)$ *if and only if there exists an (n,k)-PVM $\{e_{x,a} : 1 \le x \le n, 1 \le a \le k\} \subseteq \mathcal{R}^\omega$ such that*
$$p(a,b|x,y) = \tau_\omega(e_{x,a}e_{y,b}).$$
.

Statement (1) has been given a more direct proof in [12]. In contrast to (2), it is known that for $n \ge 4, k \ge 3$ that $C_q(n,k) \subsetneq C_{qs}(n,k)$, see [**?**]. The proof of (3) uses the deep theory of *amenable traces* developed by Kirchberg. For more on amenable traces see the work of Musat-Rordam.

References

[1] Kenneth Dykema and Vern I. Paulsen, Synchronous correlation matrices and Connes' embedding conjecture, arXiv:1503.07207, Journal of Mathematical Physics, 57.

[2] Kenneth Dykema, Vern I. Paulsen, and Jitendra Prakash, Non-closure of the set of quantum correlations via graphs, arXiv preprint, arXiv:1709.05032, *Communications in Mathematical Physics* (2019), doi: 10.1007/s00220-019-03301-1

[3] T. Fritz, *Tsirelson's problem and Kirchberg's conjecture*, Rev. Math. Phys. 24 (2012), 1250012, 67.

[4] J.W. Helton, K.P. Meyer, Vern I. Paulsen, and Matt Satriano, Algebras, Synchronous Games and Chromatic Numbers of Graphs, New York J. Math 25(2019), 328-361.

[5] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen, MIP*=RE, arXiv:2001.04383

[6] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, and R. F. Werner, *Connes' embedding problem and Tsirelson's problem*, J. Math. Phys. 52 (2011), 012102, 12.

[7] . Se-Jin Kim, Vern I. Paulsen, and Christopher Schafhauser, A synchronous game for binary constraint systems, arXiv:1707.01016, *Journal of Mathematical Physics*, 59, 032201(2018); doi: 10.1063/1.4996867.

[8] Eberhard Kirchberg, Discrete groups with Kazhdan's property T and factorization property are residually finite, Math. Ann., 299, 35–63, 1994

[9] N. Ozawa, *About the Connes embedding conjecture–algebraic approaches*, Jpn. J. Math., 8:147-183, 2013.

[10] V. I. Paulsen, S. Severini, D. Stahlke, I. G. Todorov, and A. Winter, *Estimating quantum chromatic numbers*, J. Funct. Anal. 270 (2016), no. 6, 2188?2222.

[11] William Slofstra,

[12] . Thomas Vidick, Almost synchronous quantum correlations arXiv:2103.02468

[13] Dan Voiculescu, Free entropy, Bull. London Math. Soc., 34(2002) 257–278.

Institute for Quantum Computing and Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada N2L 3G1

Email address: `vpaulsen@uwaterloo.ca`