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Abstract. These notes are intended to accompany and parallel my lectures at Copenhagen. These notes go into more detail than I will be able to provide in the lectures. They assume some background in operators on a HIlbert space. Most of this background is available in notes distributed earlier.

1. Two Person Cooperative Games

The types of games that we shall be interested in are *two person games*, which are *cooperative* and *memoryless*. Generally, the two players are referred to as Alice and Bob. Intuitively, in such a game the two players are playing cooperatively to give correct pairs of answers to pairs of questions posed by a third party often called the *Referee* or *Verifier*. Whether the pair of answers returned by the players is satisfactory or not depends not just on the individual answers but on the 4-tuple consisting of the question-answer pair.

Such a game is described by two input sets I_A, I_B, two output set O_A, O_B, and a function

$$\lambda : I_A \times I_B \times O_A \times O_B \to \{0, 1\},$$

often called the *rules* or *verification function*, where

$$W := \{(x, y, a, b) : \lambda(x, y, a, b) = 1\},$$

is the set of *correct* or *winning* 4-tuples and

$$L := \{(x, y, a, b) : \lambda(x, y, a, b) = 0\},$$

is the set of *incorrect* or *losing* 4-tuples.

For each *round* of the game Alice and Bob receive an input pair (x, y) and return an output pair (a, b) is referred to as a *round* of the game.

When we say that the players are not allowed to *communicate* or that the game is *non-communicating*, this means that Alice must return her answer without knowing the question y that Bob was asked and without knowing the answer b that Bob gave. Similarly, Bob does not know Alice's question-answer pair.

Thus, a game G is specified by $(I_A, I_B, O_A, O_B, \lambda)$. Before the game begins Alice and Bob have all of the above information, including knowing the function λ. Even though Alice and Bob are not allowed to communicate during the game they are allowed to communicate before the game and decide on some type of strategy.

A *deterministic strategy* for such a game is a pair of functions $f : I_A \to O_A,\ g : I_B \to O_B$ such that whenever Alice and Bob receive input pair (x, y) they respond with output pair $(f(x), g(y))$.

When we want to talk about the probability of winning such a game we also need to specify a probability density on inpuyt pairs, i.e., a function $\pi : I_A \times I_B \to [0, 1]$ such that

$$\sum_{x \in I_A, y \in I_B} \pi(x, y) = 1.$$

For games with densities, Alice and Bob also know the density before the start of the game.

Here is a very simple example:

1.1. **The CHSH Game.** Here $I_A = I_B = O_A = O_B = \mathbb{Z}_2$–the binary field. The function λ is most easily described by saying that given the input pair (x, y) they win if their output pair (a, b) satisfies

$$a + b = xy.$$

Let's say that they also know that the probabilty is given by $\pi(x, y) = 1/4$, i.e., the uniform distribution on the 4 possible input pairs.

If they choose the deterministic strategy of always returning 0 no matter what input they receive then $a + b = 0$ and so they will win unless the input pair was $(1, 1)$. Thus the expected value of this deterministic strategy is $3/4$.

Problem 1.1. *Show that among all deterministic strategies, this is the one with the greatest expected value and that there is exactly one other deterministic strategy with value 3/4.*

Problem 1.2. *Suppose that we fix $0 < t \leq 1$ and change the input probability to*

$$\pi(0, 0) = \pi(1, 0) = \pi(0, 1) = t/3,\ \pi(1, 1) = 1 - t.$$

What can you say about the best deterministic strategy in this case? (Hint: It depends on t.)

However, these games are *memoryless*, that is, if Alice and Bob receive the same input pair (x, y) at two different rounds of the game, then there is no penalty if they return different pairs at different rounds.

This allows for the possibility of strategies that produce the answer pairs randomly.

A *random strategy* for such a game yields a *conditional probability density*,

$$p(a, b|x, y),\ x \in I_A,\ y \in I_B,\ a \in O_A,\ b \in O_B,$$

which gives the conditional probability that Alice and Bob return output pair (a, b), given that they received input pair (x, y).

It is known for this CHSH game that any conditional probability produced by any "classical" type of randomness can do no better than an expected value of $3/4$, i.e., the same as without randomness.

The idea behind a *quantum strategy* for such a game is that Alice and Bob have separated labs but share a, possibly entangled, state and have measurement systems for each input. Imagine for example that there is a pair of laser beams, one shining into Alice's lab and one into Bob's.

When they receive an input pair, they each conduct the corresponding pair of measurements and receive an output a and b which they each report to the Referee.

It is known that in this case, by having beams that are entangled in just the right way and performing just the right measurements, they can increase their expected value of winning to

$$cos^2(\pi/8) \simeq .85.$$

However, there are several different mathematical models for describing the densities that they could obtain via a quantum strategy.

Whether or not the densities given by various pairs of these models are the same or not is a problem first posed by *Tsirelson*. Thanks largely to research on non-local games, we now know that all of these models yield different sets of probability densities.

Before describing what these models are, let me just say that the sets of densities given by these different models will be denoted by C_q, C_{qs}, C_{qa} and C_{qc}. Tsirelson proved that in certain setting all of these models gave rise to the same sets of densities and wondered if this was true more generally.

Work of Junge et al [16] and Ozawa [24] proved that whether or not two of these models, C_{qa} and C_{qc}, gave the same densities or not is equivalent to the famous *Connes' Embedding Problem(CEP)* having an affirmative answer. In MIP*=RE [15] the authors use non-local games to prove that $C_{qa} \neq C_{qc}$, i.e., that these sets of densities are different and that hence CEP has a negative answer.

They did this by producing a game with a perfect strategy in C_{qc} but no perfect strategy in C_{qa}.

A strategy(or its corresponding density) is called *perfect*, if the probability that it returns a wrong answer is 0. That is, it is perfect if

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0.$$

Note that a pair of functions $f : I_A \to O_A$, $g : I_B \to O_B$ yields a perfect deterministic strategy if and only if

$$\lambda(x, y, f(x), g(y)) = 1, \forall x \in I_A, y \in I_B.$$

Note that a perfect strategy has value 1 no matter what the density π is on inputs.

Earlier, Slofstra showed the first difference between two of these sets by showing that $C_q \neq C_{qa}$. He did this by producing a game that he could prove had a perfect strategy in C_{qa} but not in C_q. Similarly, the paper MIP*=RE proved $C_{qa} \neq C_{qc}$ by producing a game with a perfect strategy in C_{qc} but not in C_{qa}. Thus, the theory of perfect strategies for these games is a very powerful tool.

We now look at a special family of games.

A game $G = (I_A, I_B, O_A, O_B, \lambda)$ is called *synchronous* provided that $I_A = I_B = I$, $O_A = O_B = O$ and

$$\lambda(x, x, a, b) = 0, \ \forall x \in I, \ \forall a \neq b.$$

Thus, in a synchronous game, if in a given round, Alice and Bob are asked the same question, then they must give the same reply. We shorten the notation for a synchronous game to $G = (I, O, \lambda)$.

For the first few lectures we will focus on the problem of whether or not perfect strategies exist for games. For this reason we will not refer to densities on. inputs. In later lectures, we will return to the expected values of games, and then we will need to specify densities on inputs.

Note that if a pair of functions $f, g : I \to O$ yields a perfect deterministic strategy for a synchronous game, then necessarily $f = g$.

If a density $p(a, b|x, y)$ is a perfect density for a synchronous game then it must be a synchronous density.

Here are a few of my favorite synchronous games.

1.2. **Graph Colouring Games.** By a *graph* I mean that we have a non-empty set V called the *vertex* set and a subset $E \subseteq V \times V$ called the *edge* set satisfying:

- $(x, x) \notin E$, $\forall x \in V$(no loops),
- $(x, y) \in E \implies (y, x)) \in E$(undirected).

Such a graph is denoted by the pair $G = (V, E)$, pairs of vertices such that $(x, y) \in E$ are called *adjacent.*. By a (vertex) *k-coloring* of G, I mean a function $f : V \to \{1, ..., k\}$ such that $(x, y) \in E \implies f(x) \neq f(y)$, i.e., adjacent vertices must be assigned different colors. The smallest integer k for which a k-coloring of G exists is called the *chromatic* or *coloring* number of G and is dentoed $\chi(G)$.

The *k-coloring game* for G, denoted $Col(G, k)$ is the game with input set $I = V$ output set $O = \{1, ..., k\}$ and rules $\lambda : V \times V \to \{0, 1\}$ given by

- $\lambda(x, x, a, b) = 0, \forall x, a, b$,i.e., whenever Alice and Bo recieve the same vertex they must return the same color,
- $\forall (x, y) \in E, \ \lambda(x, y, a, a) = 0, \forall z$, i.e., whenever Alice and Bob recieve adjacent vertices they must return different colors,
- if $x \neq y$ and $(x, y) \notin E$ then $\lambda(x, y, a, b) = 1, \forall a, b$, i.e., if x and y are not adjacent and not equal then they can return any colors.

Problem 1.3. *Show that a perfect deterministic strategy is a function $f : V \to \{1, ..., k\}$ that is a k-coloring.*

For this reason, the game $Col(G, k)$ is often called a *prover system* for graph coloring. The least integer k for which a perfect deterministic strategy exists for $Col(G, k)$ is $\chi(G)$.

If Alice and Bob wanted to convince a Referee that a k-coloring existed for a particular graph, without revealing the actual coloring, then they could use a random strategy and as they won more and more rounds of this game, the probability that they had an actual coloring would be increasing. But since the game is memoryless, there is no requirement that they use the same color for a given vertex at each round. So the Referee could be becoming increasingly convinced that they do indeed have a coloring, while being totally flummoxed as to what the actual coloring might be.

Remarkably, perfect quantum strategies exist for $Col(G,k)$ for values $k < \chi(G)$. This leads to the notion of various *quantum chromatic numbers* for graphs. The least k for which there exists a perfect density in C_q for the game $Col(G,k)$ is called the *quantum chromatic number* of the graph and is denoted $\chi_q(G)$.

Not only can $\chi_q(G) < \chi(G)$, but the family of *Hadamard graphs* are known to have quantum chromatic numbers that are exponentially smaller than there chromatic numbers.

The Hadamard graphs Ω_N are defined as follows:

- the vertex set is all N-tuples of ± 1, so that Ω_N has 2^N vertices,
- two vertices $x + (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$ are adjacent if and only if
$$x \cdot y := \sum_{i=1}^{N} x_i y_i = 0.$$

Note that if N is odd then $x \cdot y \neq 0$ so the only interesting case is for N even.

It is known that $\chi(\Omega_N) > (1.06)^N$, but exact values of the chromatic number are only known for a few values of N. On the other hand $\chi_q(\Omega_N) = N$, for all even N.

If we consider the graph G with uncountably many vertices,
$$V = \mathbb{T}^N := \{(\lambda_1, ..., \lambda_N) : \lambda_i \in \mathbb{C}, |\lambda_i| = 1\},$$
and $(\vec{\lambda}, \vec{\mu}) \in E \iff \sum_i \lambda_i \mu_i = 0$, then [27] show that this graph also has $\chi_q(G) = N$. Very little is known about the chromatic number of these graphs.

We can also require that there be a perfect density in C_{qs}, C_{qa} or C_{qc} and these lead to $\chi_{qs}(G), \chi_{qa}(G)$ and $\chi_{qc}(G)$. It is known that $\chi_{qs}(G) = \chi_q(G)$ for every G and there are examples known for which $\chi_q(G) \neq \chi_{qa}(G)$.

1.3. **The Graph Homomorphism Game.** Given two graphs $G_i = (V_i, E_i), i = 1, 2$ a *homomorphism* from G_1 to G_2 is a function $f : V_1 \to V_2$ with the property that if $(x, y) \in E_1$ then $(f(x), f(y)) \in E_2$. We write $G_1 \to G_2$ to indicate that there exists a graph homomorphism from G_1 to G_2.

Graph homomorphisms are convenient for capturing many of the parameters studied in graph theory. For instance, if K_k denotes the *complete graph* on k vertices, i.e., every pair is an edge, then it is not hard to see that G has a k-colouring if and only if $G \to K_k$. Thus, $\chi(G)$ is the smallest k for which $G \to K_k$.

Similarly, a *clique* in G is a subset of vertices such that every pair is connected by an edge. It is not hard to see that G has a clique of size k if and only if $K_k \to G$.

A set of vertices in G is called *independent* if they contain no edges. If G^c denotes the *complement of* G, i.e., the graph with the same vertices, but with (x, y) an edge in G^c if and only if it was NOT an edge in G. Thus, G has an independent set of size k if and only if $K_k \to G^c$.

The graph homomorphism game $Hom(G_1, G_2)$ is the synchronous game with $I = V_1, O = V_2$ and

$$N = \{(x, y, a, b) : (x, y) \in E_1, (a, b) \notin E_2\} \cup \{(x, x, a, b) : x \in V_1, a \neq b\}.$$

Problem 1.4. *Show that a perfect deterministic strategy for this game is an actual graph homomorphism.*

Problem 1.5. *Show that the rule functions for $Col(G, k)$ and $Hom(G, K_k)$ are the same function. Thus, these games are the "same" game.*

We write $G_1 \stackrel{q}{\to} G_2$ to indicate that there is a perfect density for $Hom(G_1, G_2)$ in C_q, with similar definitions for $G_1 \stackrel{t}{\to} G_2$, for $t = qs, qa, qc$. As before,

$$G_1 \stackrel{qs}{\to} G_2 \iff G_1 \stackrel{q}{\to} G_2.$$

1.4. **The Graph Isomorphism Game.** Given a graph $G = (V, E)$ we define a function

$$rel : V \times V \to \{-1, 0, +1\},$$

via

$$rel(x, y) = \begin{cases} -1, & (x, y) \in E, \\ 0, & x = y, \\ +1, & (x, y) \notin E \text{ and } x \neq y \end{cases}.$$

Two graphs $G_i = (V_i, E_i), i = 1, 2$ are **isomorphic** if there is a one-to-one, onto function $f : V_1 \to V_2$ such that

$$rel(x, y) = rel(f(x), f(y)), \forall x, y \in V_1.$$

. In this case we write $G_1 \simeq G_2$.

The *graph isomorphism game* $Iso(G_1, G_2)$ is the game with input set $I = V_1$, output set $O = V_2$, and we will define it in terms of its winning set

$$W = \{(x, y, a, b) \in V_1 \times V_1 \times V_2 \times V_2 | rel(x, y) = rel(a, b)\}.$$

Problem 1.6. *Show that this is a synchronous game and that when $card(V_1) = card(V_2)$, then $f : V_1 \to V_2$ is a perfect deterministic strategy if and only if f is a graph isomorphism. More generally, show that f is a perfect deterministic strategy if and only if $card(V_1) \leq card(V_2)$ and G_1 is isomorphic to the induced subgraph of G_2 on the subset $f(V_1)$. (Some would call this an isomorphism onto the range.)*

For $t = q, qs, qa, qc$ we write $G_1 \simeq_t G_2$ if and only if there exists a perfect density for this game in C_t.

A beautiful result of [**?**] shows that $G_1 \simeq_{qc} G_2$ if and only if the number of graph homomorphisms from H to G_1 is equal to the number of graph homomorphisms from H to G_2 for all planar graphs H.

1.5. **Linear Constraint System Games.** . Recall that if p is a prime number then the set of integers modulo p, $\mathbb{Z}_p$ equipped with addition modulo p and multiplication modulo p is a *field*. The most familiar of these is the *binary field* $\mathbb{Z}_2 = \{0, 1\}$, where $1 + 1 \equiv 0$.

In particular, for each $k \neq 0$ there is $j \neq 0$ such that

$$jk \equiv 1(modp).$$

For example, in $\mathbb{Z}_{31}$, $2 \cdot 16 \equiv 32 \equiv 1(mod32)$ and so $16 = 2^{-1}$, while $-2 \equiv +29$.

Suppose that we are given a system of n linear equations in m variables over $\mathbb{Z}_p$:

$$f_i(x_1, ..., x_m) := \sum_{j=1}^{m} a_{i,j} x_j = b_i, 1 \leq i \leq n,$$

or in matrix vector notation,

$$A\vec{x} = \vec{b}.$$

There are two versions of the linear constraint system game. One is not synchronous and the other is synchronous.

First we discuss the non-synchronous game. In this game Alice is given an equation, i.e., $I_A = \{1, ..., n\}$ and Bob is given a variable, $I_B = \{1, ..., m\}$. Alice must return values for each of the variables in her equation and Bob must give a value to his variable. They win if Alice's variable values satisfy the equation that she was given and if value that Bob assigned to his particular variable is the same as the value that Alice gave to that variable.

For this game there is a group, called the *solution group* of the game. This group has a distinguished element J and it is known that this game has a perfect q-strategy if and only if this group has a finite dimensional representation that sends J^{p-1} to $-I$.

Here is the intuitive description of the synchronous game, denoted $LCS(A, b)$. The input set is $I = \{1, ..., n\}$. Suppose that for inputs Alice receives i_1 and Bob receives i_2. To win Alice must return values for each of the variables in equation i_1 that has a non-zero coefficient that satisfy $f_{i_1}(\vec{x}) = b_{i_1}$ and Bob must return values for each of the variables in equation i_2 that has a non-zero coefficient and satisfy $f_{i_2}(\vec{y}) = b_{i_2}$. In addition if variable j has a non-zero coefficient in both equations, then Alice and Bob must have given the same value to that variable, i.e., $x_j = y_j$.

This isn't quite a "game" as we've defined them since there is not a fixed output set. We fix that by making the rules that: To win, Alice and Bob must return vectors $\vec{v}, \vec{w} \in (\mathbb{Z}_p)^k$ such that:

(1) $f_{i_1}(\vec{v}) = b_{i_1}$, and $f_{i_2}(\vec{w}) = b_{i_2}$,
(2) $a_{i_1,j} = 0 \implies v_j = 0$ and $a_{i_2,j'} = 0 \implies w_{j'} = 0$

With these slightly modified rules, we can see that this is a synchronous game with output set $O = \mathbb{Z}_p^m$. Thus the output set has p^m elements.

It is known that these two version of the game are equivalent in the sense that one version has a perfect t-strategy if and only if the other version has a perfect t-strategy for $t = q, qa, qc$.

This brings up the interesting question of how does one prove that two games with such different descriptions have such similar behavior? We will see one way to approach such problems when we discuss the "algebra of a game".

Problem 1.7. *Show that for both versions, this game has a perfect deterministic strategy if and only if the system of equations has a solution.*

Remarkably, there exist systems of equations that have no actual solutions but which have perfect densities in one of the quantum correlation sets. A famous one of these is *Mermin's Magic Square.* This system of binary equations can most easily be represented as follows:

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & 1 \\ x_4 & x_5 & x_6 & 1 \\ x_7 & x_8 & x_9 & 1, \\ - & - & - & \\ 0 & 0 & 0 & \end{array}$$

where each horizontal row is supposed to 1 and each vertical column is supposed to 0.

A moments reflection shows that this system of equations has no solution. However, the LCS game has a perfect strategy in $C_q(6, 2^9)$.

Slofstra was able to prove that $C_q \neq C_{qa}$ by creating a system of roughly 200 equations, each involving 3 variables over the binary field that had a perfect density in C_{qa} but no perfect density in C_q.

There is a beautiful connection between linear system games over the binary field and graph isomorphisms.

In [1] given a binary system of equations $A\vec{X} = \vec{b}$ a graph $G_{A,b}$ is constructed with the property that $G_{A,b} \simeq G_{A,0}$ if and only the system of equations has a solution.

Moreover they prove that for $t = q, qc$,

$$G_{A,b} \simeq_t G_{A,0} \iff LCS(A, b) \text{ has a perfect density in } C_t.$$

Later it was shown that the same result holds for $t = qs, qa$ [17].

There is no reason that one needs to restrict attention to linear equations in the above analysis. Especially, over $\mathbb{Z}_2$, Boolean equations are described by non-linear equations. This gives us a small hint at how perfect quantum solutions to Boolean equations might lead to "new" logics.

2. Models for Quantum Correlations: Tsirelson's Problems

In the last section we mentioned that there were different models for quantum densities without really addressing what these models are. We remedy that problem here.

Suppose that Alice and Bob have separated, isolated labs and they can each perform one of n_A, respectively, n_B, quantum measurements and each measurement has, respectively, k_A and k_B outcomes. We let $p(a,b|x,y)$ denote the conditional probability density that Alice gets outcome a and Bob gets outcome b, when the perform measurements x and y, respectively. Such densities are also called **quantum correlations** and Tsirelson was interested in mathematical descriptions of the set of all such conditional densities.

It turns out that the axiomatic quantum theory allows for several possible mathematical descriptions of these sets of densities and Tsirelson was interested in whether these were all the same. So we start with the possible descriptions.

The basic quantum model assumes that Alice and Bob labs are described by finite dimensional state spaces, $\mathcal{H}_A, \mathcal{H}_B$ and that the state of their combined labs is given by a unit vector $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Alice's and Bob's measurements are each given by an (n,k)-POVM, $\{E_{x,a} : 1 \le x \le n_A, 1 \le a \le k_A\}$ and $\{F_{y,b} : 1 \le y \le n_B, 1 \le b \le k_B\}$, which means we have families of projections such that

$$\sum_{a=1}^{k_A} E_{x,a} = I_{\mathcal{H}_A}, \forall x, \text{ and } \sum_{b=1}^{k_B} F_{y,b} = I_{\mathcal{H}_B}, \forall y,$$

and

$$p(a,b|x,y) = \langle \psi | (E_{x,a} \otimes F_{y,b}) \psi \rangle.$$

We let $C_q(n_A, n_B, k_A, k_B)$ denote the set of all $p(a,b|x,y)$ that can be obtained as above, which we call the **quantum correlations** or **quantum densities**. Note that since $0 \le p(a,b|x,y) \le 1$ that we can always regard $C_q(n_A, n_B, k_A, k_B)$ as a subset of the compact set $[0,1]^{n_A n_B k_A k_B}$. Generally, we shall be interested in the case that $n_A = n_B = n$ and $k_A = k_B = k$, in which case we shorten this to $C_q(n,k)$.

A slightly more general model is to allow $Ḩ_A$ and $\mathcal{H}_B$ to be arbitrary Hilbert spaces in which case we denote this larger set by $C_{qs}(n_A, n_B, k_A, k_B)$ where the subscript stands for **quantum spatial**.

There is no reason that either of these sets needs to be closed. However, a nice result that uses C*-algebra theory is that they both have the same closure and we set

$$C_{qa}(n_A, n_B, k_A, k_B) := C_q(n_A, n_B, k_A, k_B)^- = C_{qs}(n_Q, n_B, k_A, k_B)^-.$$

These are called the **quantum approximate** correlations.

An even more general model is to assume that the combined state space of Alice and Bob does not decompose as a tensor product but instead that it is a single Hilbert space $\mathcal{H}$ so that they each have POVM's on this space,

$$\{E_{x,a} : 1 \le x \le n_A, 1 \le a \le k_A\} \subseteq B(\mathcal{H}),\ \{F_{y,b} : 1 \le y \le n_B, 1 \le b \le k_B\} \subseteq B(\mathcal{H}),$$

with the property that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a},\ \forall x, y, a, b$. We call this a commuting model. Note that it is only Alice's operators that must commute with Bob's operators. There is no requirement that Alice's operators commute among themselves for different inputs.

The set of all

$$p(a, b|x, y) = \langle \phi | E_{x,a}F_{y,b}\phi \rangle,$$

that can be obtained in this manner for some commuting model and some unit vector ϕ is denoted $C_{qc}(n_A, n_B, k_A, k_B)$ and called the **quantum commuting** correlations. This set is known to be closed but the proof needs some C*-algebra theory

The explanation for this commuting hypothesis is that the outcome should not depend on the order of applying their measurements. Note that in the tensor cases we have that

$$E_{x,a} \otimes F_{y,b} = (E_{x,a} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes F_{y,b}) = (I_{\mathcal{H}_A} \otimes F_{y,b})(E_{x,a} \otimes I_{\mathcal{H}_B}).$$

so it is a commuting correlation. It is known that

$$C_q(n_A, n_B, k_A, k_B) \subseteq C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{qa}(n_A, n_B, k_A, k_B) \subseteq C_{qc}(n_A, n_B, k_A, k_B).$$

Most of these containments are fairly straightforward, except for $C_{qs} \subseteq C_{qa}$ which uses results about *residually finite dimensional(RFD)* C*-algebras.

In the case that $n_A = n_B = k_A = k_B = 2$, Tsirelson proved that these sets are all equal, and wondered if this could be true more generally.

At first Einstein was dubious about entanglement and attempted to explain it away with a theory of *local, hidden variables.* Essentially this theory postulates that the randomness observed in quantum measurements is occurring because there is some *hidden* probability space (Ω, P) and for each $x \in I_A, y \in I_B$ random variables

$$f_x : \Omega \to O_A,\ g_y : \Omega \to O_B,$$

such that each time an experiment is conducted one is really evaluating these random variables at some unknown point $\omega \in \Omega$, i.e., if Alice conducts measurement x and Bob measurement y, then the values of the outcomes will be $a = f_x(\omega), b = g_y(\omega)$ and the only reason that we cannot predict a priori the values of the outcome is that we do not a priori know which point ω we will be evaluating these random variables at when we perform the measurement.

Problem 2.1. *Prove that in this case*

$$p(a, b|x, y) = P(\{\omega \in \Omega | f_x(\omega) = a, g_y(\omega) = b\}).$$

The set of all conditional probability densities that can be obtained in this fashion, as we vary the probability space and the random variables are called the **local densities** and is denoted by

$$C_{loc}(n_A, n_B, k_A, k_B).$$

It is not too hard to show that

$$C_{loc}(n_A, n_B, k_A, k_B) \subseteq C_q(n_A, n_B, k_A, k_B).$$

This is what motivates the term **non-local games**, which is a bit of a misnomer. It really refers to the fact that we are allowing these two person cooperative games to be played using densities that are non-local.

Problem 2.2. *Prove that a game has a perfect local density if and only if it has a perfect deterministic strategy.*

Problem 2.3. *Prove that $p(a,b|x,y)$ is an extreme point of $C_{loc}(n_A, n_B, k_A, k_B)$ if and only if there are functions $f : \{1,...,n_A\} \to \{1,...,k_A\}$ and $g : \{1,...,n_B\} \to \{1,...,k_B\}$ such that*

$$p(a,b|x,y) = \begin{cases} 1 & \text{when } a = f(x),\ b = g(y), \\ 0 & \text{else.} \end{cases}$$

As we remarked earlier, Slofstra was the first to show that $C_q(n,k)$ was not a closed set for n,k sufficiently large. We now know that it is not a closed set for most values of n,k and that these sets are very pathological.

Theorem 2.4 (Dykema-P-Prakash[8])**.** *The sets $C_q(n,k)$ are not closed for every $n \geq 5, k \geq 2$. Let $\frac{\sqrt{5}-1}{2\sqrt{5}} \leq t \leq \frac{\sqrt{5}+1}{2\sqrt{5}}$ and for $0 \leq a,b \leq 1,\ 1 \leq x,y \leq 5$ set*

$$p(0,0|x,x) = t, \qquad p(0,1|x,x) = p(1,0|x,x) = 0, \qquad p(1,1|x,x) = 1-t,$$

and for $x \neq y$, set

$$p(0,0|x,y) = \frac{1}{4}t(5t-1), \qquad p(0,1|x,y) = p(1,0|x,y) = \frac{5}{4}t(1-t),$$
$$p(1,1|x,y) = \frac{1}{4}(1-t)(4-5t).$$

Then $p \in C_{qa}(5,2)$ for all t in this interval, but $p \in C_q(5,2)$ only for t rational.

Note that this is a nice continuous path of correlations p_t but to "decide" if p_t belongs to $C_q(5,2)$ one must be able to decide if t is rational. For example it is still unknown if $e+\pi$ is rational. So if we take a rational multiple of $e+\pi$ that lands us in the above interval, then for such values of t it is still unknown if p_t belongs to $C_q(5,2)$.

By Tsirelson's results, $C_q(2,2)$ is closed, but it is still not known if $C_q(3,2)$ and $C_q(4,2)$ are closed. Combining the above theorem with results from [17], we know that $C_{qs}(n,k)$ is not closed for every $n \geq 5, k \geq 2$. In [5] it is shown that $C_{qs}(4,3)$ is not closed.

Before leaving this section, we want to mention one more family of correlations. This is the largest set of abstract conditional densities that obeys some natural axioms from probability.

A collection of numbers $p(a,b|x,y)$ is called a **non-signalling density**(or correlation) provided that:

- $p(a,b|x,y) \geq 0, \forall x,y,a,b$
- $1 = \sum_{a,b} p(a,b|x,y), \ \forall x,y$
- $\forall x,y,y', \ \sum_b p(a,b|x,y) = \sum_b p(a,b|x,y')$. This common value is denoted $P_A(a|x)$ and is called the *conditional probability that Alice gets outcome a given input x*,
- $\forall y,x,x', \ \sum_a p(a,b|x,y) = \sum_b p(a,b|x',y)/$ This common value is denoted $p_B(b|y)$ and is called the *conditional probability that Bob gets outcome b given input y.*

The set of all non-signalling densities is denoted C_{ns}.

For all of the games mentioned above, it is also interesting to determine if they have perfect non-signalling densities.

Problem 2.5. *Prove that all of the densities in C_{qc} are non-signalling.*

3. Synchronous Densities, Traces, and the Fundamental Orthogonality Relations

Given a synchronous game $G = (I,O,\lambda)$ with $n = card(I)$ and $k = card(O)$ we see that any perfect density $p(a,b|x,y)$ for this game must satisfy

$$p(a,b|x,x) = 0, \forall a \neq b, \forall x.$$

We call densities that satisfy this property **synchronous** and we use the superscript s to denote the subset of synchronous densities. So we have $C_t^s(n,k) \subseteq C_t(n,k)$, for $t = loc, q, qs, qa, qc, ns$ and

$$C_{loc}^s(n,k) \subseteq C_q^s(n,k) \subseteq C_{qs}^s(n,k) \subseteq C_{qa}^s(n,k) \subseteq C_{qc}^s(n,k) \subseteq C_{ns}^s(n,k).$$

It turns out that such densities arise from *traces* on C*-algebras, so we need to introduce and understand this concept.

Everyone is familiar with the concept of the trace of matrices:

$$Tr : M_n \to \mathbb{C}, \ Tr((a_{i,j})) = \sum_{i=1}^n a_{i,i},$$

this has the property that if $A = (a_{i,j})$ is positive semidefinite, then $Tr(A) \geq 0$ and given any two matrices

$$Tr(AB) = Tr(BA).$$

Note that this last property implies that given any commutator,

$$[A,B] := AB - BA,$$

we have that $Tr(AB - BA) = 0$. Since the Tr is linear, it will also vanish on sums of commutators and since $Tr(I_n) = n \neq 0$ this gives us a very easy way to see that the identity matrix cannot be expressed as a sum of commutators!

An abstract **trace** on a unital C*-algebra $\mathcal{A}$ is defined to be any linear functional $\tau : \mathcal{A} \to \mathbb{C}$ such that

- $\tau(a^*a) \geq 0, \forall a \in \mathcal{A}$,
- $\tau(ab) = \tau(ba)$,
- $\tau(I_{\mathcal{A}}) = 1$, where $I_{\mathcal{A}}$ denotes the identity element.

We call the pair $(\mathcal{A}, \tau)$ a **tracial C*-algebra**.

Problem 3.1. *Show that there is a unique trace $tr_n : M_n \to \mathbb{C}$ and that it is given by*

$$tr_n(A) = \frac{1}{n} Tr(A).$$

Generally a C*-algebra can have many traces or no traces. In particular if the identity can be written as a sum of commutators, then it is impossible to have a trace, since one would need the trace of the identity to be both 1 and 0. Here are examples of a C*-algebra with no traces and one with a one parameter family of traces.

Problem 3.2. *Let $\ell^2(\mathbb{N})$ be the Hilbert space of square summable sequences. This space has an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ where e_n is the vector that is 1 in the n-th coordinate and 0 elsewhere.(In the physicists notation $e_n = |n\rangle$). Prove that the identity operator on this space is a sum of commutators and, consequently, there can be no trace on $B(\ell^2(\mathbb{N}))$. (Hint: First consider the operator that maps $e_n \to e_{2n}$.)*

Problem 3.3. *Let $\mathcal{A} \subseteq M_{n+k}$ consist of all block diagonal matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with A $n \times n$ and B $k \times k$. Prove that setting*

$$\tau(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = t tr_n(A) + (1-t) tr_k(B), \ 0 \leq t \leq 1,$$

defines a trace on $\mathcal{A}$ and that every trace is of this form.

Here is the key theorem connecting traces and synchronous densities.

Theorem 3.4 (P-Severini-Stahlke-Todorov-Winter [26])**.**

1. *$p \in C_{qc}^s(n,k)$ if and only if there exists a tracial C*-algebra $(\mathcal{A}, \tau)$ and an (n,k)-PVM $\{e_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ in $\mathcal{A}$ such that*

$$p(a, b|x, y) = \tau(e_{x,a} e_{y,b}).$$

2. *$p \in C_q(n,k)$ if and only if in the above representation we can assume that $\mathcal{A}$ is finite dimensional.*

We sketch one of the key ideas of the proof. Suppose that we have written

$$p(a, b|x, y) = \langle \phi | E_{x,a} F_{y,b} \phi \rangle,$$

then

$$1 = \sum_{a,b=1}^{k} p(a,b|x,x) = \sum_{a=1}^{k} p(a,a|x,x) = \sum_{a=1}^{k} \langle E_{x,a}\phi | F_{x,a}\phi\rangle \le$$
$$\sum_{a=1}^{k} \|E_{x,a}\phi\| \cdot \|F_{x,a}\phi\| \le (\sum_{a=1}^{k} \|E_{x,a}\phi\|^2)^{1/2} (\sum_{a=1}^{k} \|F_{x,a}\phi\|^2)^{1/2} = 1.$$

Thus, the inequality is an equality and this in turn implies that $E_{x,a}\phi = F_{x,a}\phi, \forall x, a$.

Using this one shows that if we let $\mathcal{A}$ be the C*-algebra generated by $\{E_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ and let $\tau : \mathcal{A} \to \mathbb{C}$ be the state given by $\tau(X) = \langle \phi | X\phi\rangle$, then $\tau(XY) = \tau(YX)$, i.e., τ is a trace.

This proves one direction of (1). The converse, that setting $p(a,b|x,y) = \tau(E_{x,a}E_{y,b})$ when τ is a trace defines an element of $C_{qc}(n,k)$ is a standard argument for experts in C*-algebras.

Note that if $p \in C_q^s(n,k)$ then the $E_{x,a}$ would all be matrices and so $\mathcal{A}$ would be a finite dimensional C*-subalgebra of this matrix algebra. However, this does not imply that τ is the usual trace, unlike full matrix algebras where the trace is unique, subalgebras can have many traces.

Problem 3.5. *Complete the proof that τ as defined above satisfies $\tau(XY) = \tau(YX)$. (HInt: FIrst consider words in the generators.)*

The characterization of C_{qs}^s and C_{qa}^s came later and was quite a bit harder. We mention that in the Connes Embedding Problem there is one special tracial C*-algebra that plays a central role. It is denoted $(\mathcal{R}^\omega, \tau_\omega)$. For those interested in a bit more background, see the supplemental notes on CEP. For the rest of us, it is enough to know that this is just some important tracial C*-algebra.

Theorem 3.6 (Kim-P-Schafhauser)**.** *(1) $C_{qs}^s(n,k) = C_q^s(n,k), \ \forall n,k,$*

(2) $p(a,b|x,y) \in C_{qa}^s(n,k)$ if and only if there exists an (n,k)-PVM $\{e_{x,a} : 1 \le x \le n, 1 \le a \le k\} \subseteq \mathcal{R}^\omega$ such that

$$p(a,b|x,y) = \tau_\omega(e_{x,a}e_{y,b}).$$

Problem 3.7. *Note that if $p(a,b|x,y) \in C_{qc}^s(n,k)$ then*

$$p(a,b|x,y) = \tau(e_{x,a}e_{y,b} = \tau(e_{y,b}e_{x,a}) = p(b,a|y,x).$$

If $p(a,b|x,y) \in C_{ns}^s(n,k)$, then does it follow that $p(a,b|x,y) = p(b,a|y,x)$? Prove or give a counterexample.

Here is the key definition and theorem about perfect strategies for synchronous games.

Given a synchronous game $G = (I, O, \lambda)$ with $card(I) = n$ and $card(O) = k$ we say that an (n,k)-PVM, $\{E_{x,a} : x \in I, a \in O\}$ satisfies the **fundamental orthogonality relations(FOR)** for the game if and only if

$$\lambda(x,y,a,b) = 0 \implies E_{x,a}E_{y,b} = 0.$$

Theorem 3.8 (Helton-Meyer-P-Satriano [13], Kim-P-Schafhauser [17]). *Let $G = (I, O, \lambda)$ be a synchronous game. Then G has a perfect strategy in:*

1. *C_{qc} if and only if there is an (n, k)-PVM satisfying the FOR in a tracial C^*-algebra,*
2. *C_q if and only if there is an (n, k)-PVM satisfying the FOR in a matrix algebra,*
3. *C_{qa} if and only if there is an (n, k)-PVM satisfying the FOR in $\mathcal{R}^\omega$.*

Let's see what these relations are for a few games. First note that since every game is synchronous,

$$\lambda(x, x, a, b) = 0, a \neq b \implies E_{x,a}E_{x,b} = 0,$$

i.e., for each x, $\{E_{x,a} : 1 \leq a \leq k\}$ is an orthogonal family of projections summing to the identity. Since these relations hold for every game, I will often only mention the "extra" orthogonality relations.

3.1. **The Graph Colouring Game.** Given a graph $G = (V, E)$ we see that the only extra relations are that

$$(x, y) \in E \implies Ex, aE_{y,a} = 0, \ \forall a.$$

Problem 3.9. *Suppose that we try to n-colour the complete graph on $n+1$ vertices. Write down the equations that must be satisfied. Show that it is impossible for these FOR to be satisfied by a set of operators on a Hilbert space. Conclude that this game cannot have a perfect t-strategy for $t = loc, q, qs, qa, qc$(Hint: Consider $\sum_a \sum_x E_{x,a}$ and $\sum_x \sum_a E_{x,a}$.)*

Problem 3.10. *Does the above game have a perfect ns-strategy?*

3.2. **The Graph Isomorphism Game.** . Given graphs $G_i = (V_i, E_i), i = 1, 2$ the rules imply that

$$rel(x, y) \neq rel(a, b) \implies E_{x,a}E_{y,b} = 0.$$

These were analyzed in the paper [1] where it was shown that these relations are equivalent to the following conditions:

1. For each $x \in V_1$, $\{E_{x,a} : a \in V_2\}$ is an orthogonal family of projections summing to the identity.
2. For each $a \in V_2$, $\{E_{x,a} : x \in V_1\}$ is an orthogonal family of projections summing to the identity.
3. For each $x \in V_1$ and $a \in V_2$,

$$\sum_{\{x_1:(x,x_1)\in E_1\}} E_{x_1,a} = \sum_{\{a_2:(a,a_2)\in E_2\}} E_{x,a_2}.$$

This last relation is best visualized as follows. If we let A_G denote the **adjacency matrix** of a graph, i.e., the matrix that is 1 in the (x_1, x_2) entry if and only if $(x_1, x_2) \in E$ and let $(E_{x,a})$ denote the matrix of projections that has the projection $E_{x,a}$ in its $x, a)$ entry, then the third relation is that

$$A_{G_1}(E_{x,a}) = (E_{x,a})A_{G_2}.$$

For another direct derivation of these relations in language closer to these notes see [3].

3.3. **Linear System Games.** Given a system of $n \times m$ equations $A\vec{x} = \vec{b}$ over $\mathbb{Z}_p$ the extra FOR equations can be summarized as follows:

(1) $E_{i,\vec{x}} = 0$ unless $\vec{x}$ satisifes $\sum_j a_{i,j}x_j = b_i$ and $a_{i,j} = 0 \implies x_j = 0$.
(2) Given $i \neq j$ and vectors $\vec{x}$ and *vecy* satisfying (1) for i and j, respectively,
$$E_{i,\vec{x}}E_{j,\vec{y}} = 0 \text{ unless } a_{i,k}a_{j,k} \neq 0 \implies x_k = y_k.$$

We now sketch the proof of the above theorem. First we need the concept of a *faithful trace*. A tracial state $\tau : \mathcal{A} \to \mathbb{C}$ is called **faithful** provided that $\tau(x^*x) = 0 \implies x = 0$.

If a tracial state is not faithful, then $J := \{x \in \mathcal{A} : \tau(x^*x) = 0\}$ can be shown to be a 2-sided ideal and so we may form a quotient C*-algebra $\mathcal{A}/J$ and the functional $\tilde{\tau}(x + J) = \tau(x)$ is well-defined and faithful trace on this quotient C*-algebra. From this we can always reduce to the case where our density
$$p(a, b|x, y) = \tau(E_{x,a}E_{y,b}),$$
is given by a faithful trace. Next note that
$$\lambda(x, y, a, b) = 0 \implies 0 = \tau(E_{x,a}E_{y,b}) = \tau((E_{x,a}E_{y,b})^*(E_{x,a}E_{y,b})) \implies E_{x,a}E_{y,b} = 0,$$
provided that the trace is faithful.

Thus, the (n, k)-PVM must satisfy the FOR.

This proves (1).

To prove (2) one notes that the quotient of a finite dimensional C*-algebra is still finite dimensional. But every finite dimensional C*-algebra is a direct sum of matrix algebras and a trace on a direct sum of matrix algebras is just a convex combination of the (normalized) trace on each matrix summand. Now show that the restriction of the projections to each summand must satisfy the FOR.

Statement (3) follows from the fact that the projections can be taken to be in $\mathcal{R}^\omega$ and the fact that the trace τ_ω is known to be faithful.

Problem 3.11. *Prove that every trace on a direct sum of matrix algebras is a convex combination of the normalized trace on each matrix summand. Which convex combinations are faithful? Now show, the last claim, that if a POVM in a direct sum satisfies the FOR then each the restriction to each block satisfies the FOR.*

The game in MIP*=RE that has a perfect strategy in C_{qc} but not in C_{qa} is actually a synchronous game! So the perfect strategy in C_{qc} must actually be a synchronous strategy and so we have that there is a tracial C*-algebra $(\mathcal{A}, \tau)$ containing a (n, k)-PVM that satisfies the FOR of the game. But since their game has no perfect strategy in C_{qa} there cannot exist a (n, k)-PVM in $\mathcal{R}^\omega$ satisfying these FOR.

Corollary 3.12. *By MIP*=RE, there exists a finite set of orthogonality relations*

$$\lambda(x, y, a, b) = 0 \implies E_{x,a}E_{y,b} = 0,$$

that can be realized in a tracial C-algebra* $(\mathcal{A}, \tau)$ *but there are no projections in* $\mathcal{R}^\omega$ *satisfying these relations. Hence, there is no trace preserving *-homomorphism from* $(\mathcal{A}, \tau)$ *into* $(\mathcal{R}^\omega, \tau_\omega)$*.*

CEP asks for a trace preserving *-homomorphism of $(\mathcal{A}, \tau)$ into $(\mathcal{R}^\omega, \tau_\omega)$ for every tracial C*-algebra. The above shows that there can be no *-homomorphism of $\mathcal{A}$ into $\mathcal{R}^\omega$ and that the obstruction is just some FOR. This is a stronger negation of the CEP.

4. Group Algebras and the Algebra of a Synchronous Game

A *unitary representation* of a group G is a group homomorphism from G into the group of unitaries on some Hilbert space. Since unitaries satisfy $U^{-1} = U^*$, if $\rho : G \to B(\mathcal{H})$ is a unitary representation, then $\rho(g^{-1}) = \rho(g)^*$.

If we let $\sigma_k = (\mathbb{Z}_k, +)$ denote the cyclic group of order k, then every unitary representation of this group is determined by a unitary U with $U^k = I_{\mathcal{H}}$ such that $\rho(j) = U^j$. Since $U^k = I$ it is easy to see that every eigenvalue of U must be a k-th root of unity. If we set $\omega = e^{2\pi i/k}$, then the projection E_j onto the eigenspace for ω^j is given by

$$E_j = \sum_{r=0}^{k-1} (\omega^{-j}U)^r,$$

and

$$U = \sum_{j=0}^{k-1} \omega^j E_j.$$

One key point here is that the eigenprojections are not elements of the group but are linear combinations of group elements.

This is one of the motivations for studying *group algebras*. Given a group G, the complex group algebra, denoted $\mathbb{C}(G)$ is a vector space with basis $\{u_g : g \in G\}$. We use u_g for the basis elements instead of just g as a reminder that these elements should correspond to unitaries. We define a product by the rule $u_g \cdot u_h = u_{gh}$. Thus, given $a = \sum_i \alpha_i u_{g_i} \in \mathbb{C}(G)$ and $b = \sum_j \beta_j u_{h_j} \in \mathbb{C}(G)$ where $\alpha_i, \beta_j \in \mathbb{C}$, we have that

$$a \cdot b = (\sum_i \alpha_i u_{g_i}) \cdot (\sum_j \beta_j u_{h_j}) = \sum_{i,j} (\alpha_i \beta_j) u_{g_i h_j}.$$

This makes it the case that whenever $\rho : G \to B(\mathcal{H})$ is a unitary representation, then definiing $\tilde{\rho} : \mathbb{C}(G) \to B(\mathcal{H})$ by

$$\tilde{\rho}(\sum_i \alpha_i u_{g_i}) = \sum_i \alpha_i \rho(g_i),$$

defines an algebra homomorphism.

Another motivation comes from the solution group approach to the non-synchronous version of the linear constraint system game. Recall that in this case there was a group, which I will call $SG_{(A,b)}$ with a distinguished element J and one wanted to study representations of this group that sent J^{p-1} to $-I$. Note that $J^{p-1}+I$ is not an element of group but is in the group algebra. Thus, the solution group approach really wants to study represents of the quotient of the group algebra $\mathbb{C}(SG_{(A,b)})$ by the ideal generated by $J^{p-1}+I$, denoted $\mathbb{C}(SG_{(A,b)})/ < J^{p-1}+I >$ and it is properties of this algebra that determines existence of perfect strategies of the various types.

Returning to general group algebras, recall that the u_g are like placeholders for a unitary, so it is natural to define $u_g^* := u_{g^{-1}}$ and extend this to $\mathbb{C}(G)$ be setting

$$(\sum_i \alpha_i u_{g_i})^* := \sum_i \overline{\alpha_i} u_{g_i^{-1}}.$$

Now if ρ is a unitary representation, then $\tilde{\rho}$ is also a *-homomorphism, i.e.,

$$[\tilde{\rho}(\sum_i \alpha_i u_{g_i})]^* = \tilde{\rho}((\sum_i \alpha_i u_{g_i})^*).$$

This makes $\mathbb{C}(G)$ into what is known as a **-algebra.*

Conversely, one can show that if $\gamma : \mathbb{C}(G) \to B(\mathcal{H})$ is a *-homomorphism then setting $\rho(g) = \gamma(u_g)$ defines a unitary representation of G and $\gamma = \tilde{\rho}$.

In summary, studying unitary representations of groups is the same as studying *-homomorphisms of the corresponding group algebra. The group algebra is often more convenient because it contains linear combinations of group elements which can't be discussed in the context of groups.

For the cyclic group of order k, $\sigma_k = (\mathbb{Z}_k, +)$, we see that $\mathbb{C}(\sigma_k)$ has two natural bases, $\{u_0, ..., u_{k-1}\}$ and $\{e_0, ..., e_{k-1}\}$ where

$$e_j := \sum_{r=0}^{k-1} (\omega^{-j} u_1)^r = \sum_{r=0}^{k-1} \omega^{-jr} u_r,$$

represent the projections onto the eigenspaces of u_1.

Given a k-PVM $\{E_0, ..., E_{k-1}\}$ we see that setting $U = \sum_{j=0}^{k-1} \omega^j E_j$ defines a unitary of k.

Next we want to see what studying a (n, k)-PVM corresponds to doing. For this we need the concept of the *free product* of groups.

Given two groups G, H, twe would like to form a group that contains both G and H as subgroups, but in such a way that there is no assumption that the elements of G commute with the elements of H. This is achieved by their free product, denoted $G \star H$. This group consists of all possible *words* in an alphabet consisting of the set $G \cup H$. Thus,

$$G \star H = \{g, h, g_1 \star h_1, h_2 \star g_2, g_3 \star h_3 \star g_4, ...\}.$$

The rule for multiplying two such words is called *concatenation.* If two words end in letters from different groups then their concatenation is to just form the longer word. For example,

$$(g_1 \star h_1) \star (g_2 \star h_2 \star g_3) = g_1 \star h_1 \star g_2 \star h_2 \star g_3.$$

On the other hand if the first word ends with an element of the same group as the second word starts with, then we just multiply those elements. For example,

$$(g_1 \star h_1) \star (h_2 \star g_2) = g_1 \star (h_1 h_2) \star g_2.$$

Finally, the inverse is defined by taking the inverse of each element in the reverse order, e.g.,

$$(g_1 \star h_1 \star g_2)^{-1} = g_2^{-1} \star h_1^{-1} \star g_1^{-1}.$$

The reason for all this fuss and bother is the following universal property of the free product: Given groups G, H, K and homomorphisms, $\rho : G \to K$ and $\pi : H \to K$ there is a unique homomorphism

$$\gamma : G \star H \to K \text{ with } \gamma(g \star h) = \rho(g)\pi(h).$$

The homomorphism γ is generally denoted $\rho \star \pi$.

Finally, we should note that $G \star H = H \star G$ and $\rho \star \pi = \pi \star \rho$ and that the identity elements satisfy $e_G = e_H = e_G \star e_H = e_{G\star H} = (g \star h) \star (h^{-1} \star g^{-1})$ and many others.

Why this all matters to us is the following. Given an (n,k)-PVM $\{E_{x,a} : 1 \le x \le n, 0 \le a \le k-1\}$ we get unitaries,

$$U_x = \sum_{a=0}^{k-1} \omega^a E_{x,a},$$

and each unitary corresponds to a representation of the cyclic group of order k. So these n unitaries correspond to a representation of the free product of n copies of the cyclic group of order k. We denote this group by $\mathbb{F}(n,k)$.

Thus, we have one group algebra $\mathbb{C}(\mathbb{F}(n,k))$ so that studying it is the same as studying (n,k)-PVM's. Moreover, since synchronous densities are the same as studying various kinds of traces on the algebra generated by Alice's (n,k)-PVM, we see that studying synchronous densities is closely related to the study of traces on the group algebra $\mathbb{C}(\mathbb{F}(n,k))$. In particular, as generators for this algebra we can take either n unitaries $u_x, 1 \le x \le n$ or their corresponding spectral projections, $\{e_{x,a} : 1 \le x \le n, 0 \le a \le k-1\}$.

Given a synchronous game, $G = (I, O, \lambda)$ with n inputs and k outputs, we know that to have a perfect strategy we will need an (n,k)-PVM, which means that basically, we are starting with the algebra $\mathbb{C}(\mathbb{F}(n,k))$ such that the generators $\{e_{x,a}\}$ satisfy the FOR. The way that we can algebraically create an algebra that satisfies the FOR is to take all the products that we want to be 0, form the 2-sided ideal that they generate and take a quotient. However, we also want to preserve the *-structure, so for this reason we take the 2-sided *-ideal.

So we let the *ideal of the game* be the 2-sided ideal, denoted $\mathcal{I}_G$ generated by the set of elements,

$$\{e_{x,a}e_{y,b} : \lambda(x,y,a,b) = 0\} \cup \{e_{y,b}e_{x,a} : \lambda(x,y,a,b) = 0\}.$$

A typical element of $\mathcal{I}_G$ has the form,

$$\sum_i p_i e_{x_i,a_i} e_{y_i,b_i} q_i,$$

where p_i, q_i are arbitrary elements of $\mathbb{C}(\mathbb{F}(n,k))$ and either $\lambda(x_i, y_i, a_i, b_i) = 0$ or $\lambda(y_i, x_i, b_i, a_i) = 0$.

We define the **algebra of the game** to be the *-algebra that is the quotient,

$$\mathcal{A}(G) := \mathbb{C}(\mathbb{F}(n,k))/\mathcal{I}_G.$$

If we let $\widehat{e_{x,a}} := e_{x,a} + \mathcal{I}_G$ denote the image of $e_{x,a}$ in the quotient then these elements generate $\mathcal{A}(G)$ and satisfy:

(1) $\lambda(x,y,a,b) = 0 \implies \widehat{e_{x,a}}\widehat{e_{y,b}} = 0$,
(2) $\widehat{e_{x,a}}^2 = \widehat{e_{x,a}}^* = \widehat{e_{x,a}}$,
(3) $\sum_a \widehat{e_{x,a}} = \widehat{1}$.

Restating our theorem about the FOR we have:

Theorem 4.1. [13] *Let G be a synchronous game.*

(1) *G has a perfect deterministic strategy if and only if G has a perfect loc-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ to $\mathbb{C}$.*
(2) *G has a perfect q-strategy if and only if G has a perfect qs-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ to some matrix algebra.*
(3) *G has a perfect qa-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ into $\mathcal{R}^\omega$.*
(4) *G has a perfect qc-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ into some tracial C*-algebra.*

It is possible, and in fact happens quite often, that the identity element belongs to $\mathcal{I}_G$ in which case $\mathcal{I}_G = \mathbb{C}(\mathbb{F}(n,k))$ and hence the quotient collapses to be 0. In this case there can be no unital *-homomorphisms into anything with a unit and hence we know that these games have no perfect strategies of any flavor. This often gives us a simple algebraic test to show that the game cannot have a perfect strategy of any flavour.

Problem 4.2. *Let $G = Col(K_3, 2)$ the game for 2-colouring the complete graph on three vertices. Prove that $1 \in \mathcal{I}_G$.*

It is also the case that if we look at the game for 3-colouring the complete graph on four vertices, then $1 \in \mathcal{I}_G$, but this is much harder to show..

Remarkably, in [13] a machine assisted proof is given that $1 \notin \mathcal{I}_G$ when $G = Col(K_5, 4)$, i.e., the game for 4-colouring the complete graph on five vertices. This was achieved by using a non-commutative Grobner basis program. The algorithm produced so many elements for the Grobner basis that it appears unlikely that a simple direct proof can be given of this fact.

Thus, $\mathcal{A}(Col(K_5, 4)) \neq (0)$, but by an earlier exercise, one can show that this *-algebra cannot be represented on any Hilbert space, and in particular cannot have a perfect qc-strategy. Thus, the algebra $\mathcal{A}(G)$ appears to not give us a full picture of when perfect qc-strategies exist. It also shows that synchronous games can give us a method to produce quite esoteric *-algebras.

So what is $\mathcal{A}(G)$ good for? Here is some applications:

Corollary 4.3. [13] *Let $t = loc, q, qs, qa, qc$ and let G_1 and G_2 be two synchronous games. Assume that there exists a unital *-homomorphism from $\mathcal{A}(G_1)$ to $\mathcal{A}(G_2)$. If G_2 has a perfect t-strategy, then G_1 also has a perfect t-strategy.*

In particular, even if the games might seem very different, whenever $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ are *-isomorphic algebras we see that they share existence/non-existence of perfect strategies of the various flavours.

For a second application, recall that for linear systems there were two versions of the game, one synchronous and another that was not synchronous. For the synchronous version we have the algebra of the game, $\mathcal{A}(LCS(A, b))$. The theory for the non-synchronous version says that there is a group, called the *solution group* $SG_{(A,b)}$ with a distinguished element J and perfect strategies are determined by whether or not this group has a representation that sends J^{p-1} to $-I$. This means that in the representation $J^{p-1} + I = 0$ and the relevant algebra to study is the quotient of the group algebra by this relation, $\mathbb{C}(SG_{(A,b)})/ < J^{p-1} + I >$.

Theorem 4.4 (A. Goldberg [10])**.** *Given a linear system of equations $A\vec{x} = \vec{b}$ over the field $\mathbb{Z}_p$, the two algebras $\mathcal{A}(LCS(A, b))$ and $\mathbb{C}(SG_{(A,b)})/ < J^{P-1} + I >$ are unitally *-isomorphic.*

Thus, we finally have a language that allows us to prove that these two versions of the game are really formally identical.

Theorem 4.5 (S. Harris [?])**.** *Let G_1 be a synchronous game. Then there is a synchronous game G_2 with a 3 element output set such that $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ are unitally *-isomorphic.*

This shows, for example, that the synchronous game constructed in MIP*=RE that has a perfect qc-strategy but no perfect qa-strategy, can be assumed, without loss of generality, to be a 3-output game.

It is possible, and in fact happens quite often, that the identity element belongs to $\mathcal{I}_G$ in which case $\mathcal{I}_G = \mathbb{C}(\mathbb{F}(n, k))$ and hence the quotient collapses to be 0. In this case there can be no unital *-homomorphisms and hence no perfect strategies of any flavor.

This also brings up the issue of can we ever just by showing that $\mathcal{A}(G) \neq (0)$ deduce that we have a perfect qc-strategy? The graph colouring games show that this is not always the case. However, some important cases of this are known.

Theorem 4.6 (A. Goldberg [10])**.** *If G is either a linear system game(over any finite field) or a graph isomorphism game and $\mathcal{A}(G) \neq (0)$, then G has a perfect qc-strategy.*

The above results show that the game algebra does not always give a complete answer to which games have perfect qc-strategies. It also leaves open the question of whether or not there is some way to algebraically determine if games have perfect qc-strategies. This problem was answered recently by Adam Bene Watts, John William Helton, and Igor Klep[32].

Let $G = (I, O, \lambda)$ be a synchronous game. By the **left ideal** generated by the relations we mean the set of all finite sume

$$\mathcal{L} := \{\sum_i p_i e_{x_i,a_i} e_{y_i,b_i} : p_i \in \mathbb{C}(\mathbb{F}(n,k)),\ \lambda(x,y,a,b) = 0 \text{ or } \lambda(y,x,b,a) = 0\},$$

and by the **right ideal** generated by the relations we mean the set of all finite sums,

$$\mathcal{R} := \{\sum_i e_{x_i.a_i} e_{y_i,b_i} p_I : p_i \in \mathbb{C}(\mathbb{F}(n,k)).\ \lambda(x,y,a,b) = 0 \text{ or } \lambda(y,x,b,a) = 0\},$$

and by the **commutator space** we mean the set of all finite sums of the form

$$\mathcal{C} = \{\sum_i p_i q_i - q_i p_i : p_I, q_i \in \mathbb{C}(\mathbb{F}(n,k))\}.$$

Theorem 4.7 (Watts-Helton-Klep)**.** *Let G be a synchronous game. Then G has a perfect qc-strategy if and only if*

$$1 \notin \mathcal{L} + \mathcal{R} + \mathcal{C}.$$

This gives a completely symbolic means of determining whether or not perfect qc-strategies exist.

5. Values of Games

In order to talk about the value of a game $G = (I_A, I_B, O_A, O_B, \lambda)$ we need to also have a *prior distribution* on input pairs, i.e.,

$$\pi : I_A \times I_B \to [0,1],$$

with $\sum_{x,y} \pi(x,y) = 1$. Given a conditional probability density $p(a,b|x,y)$, the probability of winning, i.e., the *expected value* of the given strategy $p(a,b|x,y)$ is given by

$$\omega(G,\pi,p) = \sum_{x,y,a,b} \pi(x,y)\lambda(x,y,a,b)p(a,b|x,y) = \sum_{(x,y,a,b)\in W} \pi(x,y)p(a,b|x,y)).$$

Given a set S of conditional probability densities the S**-value** of the pair (G,π) is

$$\omega_S(G,\pi) := \sup\{\omega(G,\pi,p) : p \in S\}.$$

Identifying $S \subseteq [0,1]^m$, since the value is clearly a convex function of p, the value will always be attained at one of the extreme points of the closed convex hull of S.

There are many sets of conditional probability densities for which researchers attempt to compute the S-value. Among these, in particular, are the *local, quantum*, and *quantum commuting* densities.

To simplify and unify notation, we set

$$\omega_t(G,\pi) = \omega_{C_t}(G,\pi),\ \ t = loc, q, qa, qc.$$

Note that, since the value is a continuous function of the density, we have $\omega_q(G,\pi) = \omega_{qa}(G,\pi)$.

We remark that my notation is very non-standard. Generally, π is considered part of the game, so the game is just G, not (G,π). Also, the standard notation is:

$$\omega(G,\pi) = \omega_{loc}(G,\pi) \text{ and } \omega_q(G,\pi) = \omega^*(G,\pi).$$

Also since the loc densities are all convex combinations of deterministic densities, we have that $\omega_{loc}(G,\pi)$ is just the supremum over all deterministic strategies. Thus,

$$\omega_{loc}(G,\pi) = \sup\{\sum_{x,y} \pi(x,y)\lambda(x,y,f(x),g(y)) | f : I_A \to O_A,\ g : I_B \to O_B\}.$$

An often interesting question for $\omega_q(G,\pi)$ is whether or not the value is actually attained by an element of C_q. For $t = loc, qa, qc$ the value is always attained, since the corresponding sets of densities are closed and hence compact.

Computing ω and ω^* for various games has been a topic of interest in computer science for a while. These values and ω_{qc} for various games has generated a great deal of interest in the operator algebras community since it was shown by [16] and [24] that if the *Connes' embedding conjecture* had an affirmative answer, then

$$\omega_q(G,\pi) = \omega_{qc}(G,\pi),$$

for all games and densities.

Recently, [15] proved the existence of a game for which

$$\omega_q(G,\pi) < 1/2 < \omega_{qc}(G,\pi) = 1,$$

thus refuting the embedding conjecture.

For synchronous games, it is natural to restrict the allowed strategies to synchronous densities.

Given a game $G = (I,O,\lambda)$(synchronous or not) with density π we set

$$\omega_t^s(G,\pi) = \omega_{C_t^S}(G,\pi),\ t = loc, q, qc.$$

These are the values that we are interested in computing in this lecture.

Since synchronous deterministic densities correspond to a single function $f : I \to O$ we have that

$$\omega_{loc}^s(G,\pi) = \sup\{\sum_{x,y,a,b} \pi(x,y)\lambda(x,y,f(x),f(y))|f : I \to O\}.$$

Often this number is more natural than $\omega_{loc}(G,\pi)$. For an example, let's look at a graph on n vertice $G = (V,E)$ and consider the game $Col(G,2)$ with π the uniform density on edges, $\pi(x,y) = \frac{1}{|E|^2}, \forall x, y$. Thus, if Alice and Bob receive a pair (x,y) if and only if $(x,y) \in E$(recall that also $(y,x) \in E$).

To compute the synchronous value we look at all functions $f : I_A = V \to \{0,1\}$. Each function corresponds to partitioning the vertex set into two subsets, $V = S_0 \cup S_1$. Given a pair $(x,y) \in E$ we will win iff they belong to different subsets. So we would like to choose S_0, S_1 to maximize the number of edges that belong to different subsets.

This number is precisely what is meant by the **maximum cut** of G, which we denote $Max - Cut(G)$. Except that graph theorists count each edge $(x,y),(y,x)$ only once, while we count them twice.

Thus,

$$\omega_{loc}^s((Col(G,2),\pi)) = \frac{2Max - Cut(G)}{|E|^2}.$$

On the other hand if we want to compute $\omega_{loc}((Col(G,2),\pi))$ for this density, then we have two functions, $f, g : V \to \{0,1\}$. I claim that in this case,

$$\omega_{loc}((Col(G,2),\pi)) = \frac{2Max - Cut(G \times K_2)}{|E|^2},$$

where $G \times K_2$ is the tensor product of these graphs. This graph is also known as the **bipartite double cover of G**.

Thus, it is the synchronous value, not the ordinary value, that captures max-cut.

Problem 5.1. *Supply the details of these two claims.*

Problem 5.2. *Compute $\omega_{loc}^s(Col(G,2),\pi))$ and $\omega_{loc}^s(Col(G,2),\pi))$ in the case that $\pi(x,y) = \frac{1}{n^2}$ is the uniform density on all vertex pairs. (Hint: Note that in this case when $x \neq y$ and $(x,y) \notin E$, then the win is automatic.)*

We now look at the tracial characterizations of these synchronous values.

Given a C*-algebra $\mathcal{A}$ with unit, by a **trace** on $\mathcal{A}$ we mean a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ satisfying $\tau(I) = 1$, $p \geq 0 \implies \tau(p) \geq 0$ and $\tau(xy) = \tau(yx)$. The first two conditions characterize **states** on $\mathcal{A}$. When $\mathcal{A} = M_n$ the $n \times n$ matrices, it is known that there is a unique trace, namely,

$$tr_n((a_{i,j})) = \frac{1}{n}\sum_i a_{i,i} = \frac{1}{n}Tr((a_{i,j})).$$

Given a C*-algebra $\mathcal{A}$ with unit I, a **k-outcome projection valued measure(k-PVM)** is a set of k projections, $E_a = E_a^2 = E_a^*$ such that

$\sum_{a=1}^k E_a = I$. A family of n k-PVM's is a set of projections $\{E_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ with $\sum_a E_{x,a} = I, \forall x$.

Note that if $p(a,b|x,y)$ is a synchronous density, then

$$p(a,b|x,y) = \tau(E_{x,a}E_{y,b}) = \tau(E_{y,b}E_{x,a}) = p(b,a|y,x).$$

such a density is called **symmetric**.

This result translates into the following result about synchronous values.

Theorem 5.3. *Let $G = (I, O, \lambda)$ be an n input k output game and let π be a prior distribution on inputs. Then*

(1)
$$\omega_{loc}^s(G,\pi) = \sup\{ \sum_{\substack{x,y \\ (x,y,f(x),f(y))\in W}} \pi(x,y)\},$$
where the supremum is over all functions, $f : I \to O$ from inputs to outputs,

(2)
$$\omega_q^s(G,\pi) = \omega_{qa}^s(G,\pi) = \sup\{ \sum_{(x,y,a,b)\in W} \pi(x,y)tr_m(E_{x,a}E_{y,b})\},$$
where the supremum is over all families of n k-PVM's in M_m and over all m,

(3)
$$\omega_{qc}^s(G,\pi) = \sup\{ \sum_{(x,y,a,b)\in W} \pi(x,y)\tau(E_{x,a}E_{y,b})\},$$
where the supremum is over all unital C-algebras $\mathcal{A}$, traces τ, and families of n k-PVM's in $\mathcal{A}$.*

As we remarked earlier, the second supremum may not be attained.

5.1. **A Universal C*-algebra Viewpoint.** We let $\mathbb{F}(n,k)$ denote the group that is the free product of n copies of the cyclic group of order k. The full C*-algebra of this group $C^*(\mathbb{F}(n,k))$ is generated by n unitaries $u_x, 1 \le x \le n$ each of order k, i.e., $u_x^k = I$. Given any unital C*-algebra $\mathcal{A}$ with n unitaries $U_x \in \mathcal{A}, 1 \le x \le n$ of order k, there is a *-homomorphism from $C^*(\mathbb{F}(n,k))$ mapping $u_x \to U_x$. If we decompose each u_x in terms of its spectral projections,

$$u_x = \sum_{a=0}^{a-1} \alpha^a e_{x,a},$$

where $\alpha = e^{2\pi i/k}$, then $\{e_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ is a universal family of n k-PVM's, in the sense that given any set of n k-PVM's $\{E_{x,a}\}$ in a unital C*-algebra $\mathcal{A}$, there is a unital *-homomorphism from $C^*(\mathbb{F}(n,k))$ to $\mathcal{A}$ sending $e_{x,a} \to E_{x,a}$.

Values of games can be interpreted in terms of properties of the maximal and minimal C*-tensor product of this algebra with itself.

It follows from the work of [16](see also [27]) that

$p(a,b|x,y) \in C_q(n,k)^- = C_{qa}(n,k) \iff$
$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b}), \exists$ a state $s : C^*(\mathbb{F}(n,k)) \otimes_{min} C^*(\mathbb{F}(n,k)) \to \mathbb{C}$,

and that

$p(a,b|x,y) \in C_{qc}(n,k) \iff$
$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b}), \exists$ a state $s : C^*(\mathbb{F}(n,k)) \otimes C^*(\mathbb{F}(n,k)) \to \mathbb{C}$.

Given a game G and density π we set

$$P_{G,\pi} = \sum_{(x,y,a,b)\in W} \pi(x,y) e_{x,a} \otimes e_{y,b}.$$

Using the fact that norms of positive elements are attained by taking the supremum over states, we have:

Proposition 5.4. *Given an n input, k output game $G = (I,O,\lambda)$ with density π,*

$$\omega_q(G,\pi) = \|P_{G,\pi}\|_{C^*(\mathbb{F}(n,k))\otimes_{min} C^*(\mathbb{F}(n,k))},$$

and

$$\omega_{qc}(G,\pi) = \|P_{G,\pi}\|_{C^*(\mathbb{F}(n,k)\otimes_{max} C^*(\mathbb{F}(n,k)}.$$

The example of [15] gave the first proof that this minimal and maximal norms are different.

We now turn to the synchronous case.

The element $e_{x,a}e_{y,b}$ is not positive, but for any trace we have that

$$\tau(e_{x,a}e_{y,b}) = \tau(e_{x,a}e_{y,b}e_{x,a}),$$

and $e_{x,a}e_{y,b}e_{x,a} \geq 0$.

We set

$$R_{G,\pi} = \sum_{(x,y,a,b)\in W} \pi(x,y) e_{x,a}e_{y,b}e_{x,a}.$$

We also set $\mathcal{C} \subseteq C^*(\mathbb{F}(n,k))$ equal to the closed linear span of all commutators, $xy - yx$.

Given any C*-algebra $\mathcal{A}$ we let $T(\mathcal{A})$ denote the set of traces on $\mathcal{A}$ and let $T_{fin}(\mathcal{A})$ denote the set of traces that *factor through matrix algebras,* i.e., are of the form

$$\tau(a) = tr_m(\pi(a)),$$

for some m and some unital *-homomorphism $\pi : \mathcal{A} \to M_m$.

Theorem 5.5. *Let $G = (I,O,\lambda)$ be an n input, k output game with density π. Then*

(1) $\omega^s_{qc}(G,\pi) = \sup\{\tau(R_{G,\pi}) : \tau \in T(C^*(\mathbb{F}(n,k))\} = \inf\{\|R_{G,\pi} - C\| : C \in \mathcal{C}\}$,
(2) $\omega^s_q(G,\pi) = \sup\{\tau(R_{G,\pi}) : \tau \in T_{fin}(C^*(\mathbb{F}(n,k))\}$.

Two of the equalities are direct applications of the above facts. The equality of the value with the distance to the space of commutators follows from [CP79, Theorem 2.9] where it is shown that for positive elements of a C*-algebra, the supremum over all traces is equal to the distance to the space $\mathcal{C}$. We are unaware of a corresponding distance formula for the supremum over T_{fin}.

For the example of a game constructed in [15], it is known that

$$\omega_q^s(G,\pi) < 1/2 < \omega_{qc}^s(G,\pi) = 1,$$

and consequently, their results also give the first proof that $T_{fin}(C^*(\mathbb{F}(n,k))$ is not dense in $T(C^*(\mathbb{F}(n,k))$. Perhaps even more remarkable is that this difference is witnessed by the element $R_{G\pi}$ for some game, which only involves words in the generators of order three.

However, for the game of [15] is mostly given implicitly and estimates on the values of n and k to achieve their example are very large.

In summary, we see that the theory of values and synchronous values of these games gives us interesting information about C*-algebras. Thus, we are led to study these values for interesting sets of games.

6. Values and Synchronous Values of XOR Games

This section is lifted directly from [11].

In [4] quantum values of XOR games were studied extensively. In this section, we recall their results, study synchronous values of XOR games, explain how to calculate the synchronous values using semidefinite programming, and compare the two sets of results. Later, we will consider several specific examples of synchronous values of XOR games and study their properties. For XOR games the output set is always $\mathbb{Z}_2$.

Definition 6.1. A game $G = (I, \{0,1\}, \lambda)$ is an **XOR game** if there exists a function $f : I \times I \to \{0,1\}$ such that $\lambda(x,y,a,b) = 1$ if and only if $a \oplus b = f(x,y)$, where $a \oplus b$ denotes addition in the binary field.

Note that an XOR game is synchronous if and only if $f(x,x) = 0$ for all $x \in I$, and symmetric if and only if $f(x,y) = f(y,x)$.

Computing values of XOR games is especially straightforward, because of the following observation together with the Tsirelson's theory.

Proposition 6.2. *Let G be an XOR game with $|I| = n$ and prior distribution π, and let $t \in \{loc, qa, qc\}$. Then there exists a strategy $p \in C_t(n,2)$ such that $\omega_t(G,\pi) = \omega(G,\pi,p)$, where $p_A(0|x) = p_B(0|y) = 1/2$ for each $x, y \in I$.*

Proof. Since $C_t(n,2)$ is closed for each $t \in \{loc, qa, qc\}$, there exists $p \in C_t(n,2)$ such that $\omega_t(G,\pi) = \omega(G,\pi,p)$. Given such a density p, there exists a Hilbert space H, operators $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in B(H)$, and a unit vector $h \in H$ such that

$$p(0,0|x,y) = \langle P_x Q_y h, h \rangle$$

for each $x, y \in I$. For each $x \in I$, define $P_i' = P_i \oplus (I - P_i)$ and $h' = \frac{1}{\sqrt{2}}(h \oplus h)$. Let $p' \in C_t(n, 2)$ be the unique density satisfying

$$p'(0, 0|x, y) = \langle P_x' Q_y' h', h' \rangle$$

for each $x, y \in I$. Note that $p'(a, b|x, y) = \frac{1}{2}(p(a, b|x, y) + p(a \oplus 1, b \oplus 1|x, y))$. Then

$$\begin{aligned}
\omega(G, \pi, p) &= \sum_{x,y \in I, a,b \in \{0,1\}} \pi(x, y) p(a, b|x, y) \lambda(x, y, a, b) \\
&= \sum_{x,y \in I, a \oplus b = f(x,y)} \pi(x, y) p(a, b|x, y) \\
&= \sum_{x,y \in I, a \oplus b = f(x,y)} \pi(x, y) \frac{1}{2}(p(a, b|x, y) + p(a \oplus 1, b \oplus 1|x, y)) \\
&= \sum_{x,y \in I, a \oplus b = f(x,y)} \pi(x, y) p'(a, b|x, y) \\
&= \omega(G, \pi, p')
\end{aligned}$$

where we have used the fact that $a \oplus b = (a \oplus 1) \oplus (b \oplus 1)$. Since $p_A'(0|x) = p_B'(0|y) = 1/2$ and since $\omega_t(G, \pi) = \omega_t(G, \pi, p) = \omega_t(G, \pi, p')$, the statement is proven. □

Two-outcome densities satisfying $p_A(0|x) = p_B(0|y) = 1/2$ for all $x, y \in I$ are called **unbiased** densities in the literature. The following theorem is a restatement of Tsirelson's characterisation of quantum observables [**?**] in terms of unbiased densities. For those unfamiliar with the similarities and differences between quantum observables and quantum densities see [**?**, Theorem 11.8].

Theorem 6.3 (Tsirelson)**.** *Let $p(i, j|s, t)$ be a density such that $p_A(0|s) = p_B(0|t) = 1/2$ for all s, t. Then the following statements are equivalent:*

(1) *$p(i, j|s, t) \in C_{qc}(n, 2)$.*
(2) *There exist real unit vectors x_s, y_t for $1 \le s, t \le n$ such that $p(i, j|s, t) = \frac{1}{4}[1 + (-1)^{i+j} \langle x_s, y_t \rangle]$.*
(3) *$p(i, j|s, t) \in C_q(n, 2)$.*

A similar statement can be made in the synchronous case.

Theorem 6.4. *Let $p(i, j|s, t)$ be a synchronous density such that $p(0, 0|s, s) = p(1, 1|s, s)$ for all s. Then the following statements are equivalent:*

(1) *$p(i, j|s, t) \in C_{qc}^s(n, 2)$.*
(2) *There exist real unit vectors x_s for $1 \le s \le n$ such that $p(i, j|s, t) = \frac{1}{4}[1 + (-1)^{i+j} \langle x_s, x_t \rangle]$.*
(3) *$p(i, j|s, t) \in C_q^s(n, 2)$.*

Proof. Suppose the first statement is true. By Theorem 6.3, there exist unit vectors x_s, y_t for $1 \le s, t \le n$ such that $p(i, j|s, t) = \frac{1}{4}[1 + (-1)^{i+j} \langle x_s, y_t \rangle]$.

Since $p(i,j|s,s) = 0$ whenever $i \neq j$, we have $\langle x_s, y_s \rangle = 1$ for every s. By Cauchy-Schwarz, $x_s = y_s$ for every s. The other implications are straightforward. □

Remark 6.5. Given projections P_x in a C*-algebra with a trace $(\mathcal{A}, \tau)$ such that $\tau(P_x) = 1/2$, set $E_{x,0} = P_x$ and $E_{x,1} = I - P_x$. Then $\tau(E_{x,i}E_{y,j}) := p(i,j|x,y)$ is a density in C_{qc} with marginals equal to 1/2. Hence by the above result $p(i,j|x,y) \in C_q$. Give a graph $G = (V,E)$, to compute $f_{G,qc}(1/2)$ we are minimizing

$$\sum_{(x,y)\in E} \tau(P_x P_y) = \sum_{(x,y)\in E} p(0,0|x,y),$$

over all sets of projections with $\tau(P_x) = 1/2$ and, hence, $f_{G,qc}(1/2) = f_{G,q}(1/2)$. This is essentially the proof given in [8, Proposition 3.10].

We will use the theorems above, together with Proposition 6.2, to calculate the values of certain XOR games. For now, we will only provide a general formulation for these values in terms of semidefinite programs.

Remark 6.6. Let $G = (I, \{0,1\}, \lambda)$ be an XOR game with $n := |I|$, and suppose $f : I \times I \to \{0,1\}$ is a function satisfying $f(x,y) = a \oplus b$ if and only if $\lambda(x,y,a,b) = 1$ for all $a,b \in \{0,1\}$ and $x,y \in I$. Let $\pi(x,y)$ be a prior distribution on I, and let $\mathcal{G} = (G,\pi)$ denote the game G with questions asked according to the distribution π. Following [4], we define the matrix $A_{\mathcal{G}} \in M_n$ by $A_{\mathcal{G}} = ((-1)^{f(x,y)}\pi(x,y))$, which [4] call the **cost matrix**. They also study a matrix

$$B_{\mathcal{G}} := \frac{1}{2}\begin{pmatrix} 0 & A_{\mathcal{G}} \\ A_{\mathcal{G}}^T & 0 \end{pmatrix} \in M_{2n}.$$

For synchronous values, the matrix,

$$A_{\mathcal{G}}^s := \frac{1}{2}(A_{\mathcal{G}} + A_{\mathcal{G}}^T) \in M_n$$

plays a similar role to the cost matrix and we will refer to this matrix as the **symmetrized cost matrix**.

Let $\mathcal{E}_n \subseteq M_n$ denote the $n \times n$ **elliptope** defined by

$$\mathcal{E}_n := \{P \in M_n(\mathbb{R}) : diag(P) = I_n \text{ and } P \geq 0\}. \tag{1}$$

The following formula for the value of an XOR game is a restatement of results in [4]. The formula for the synchronous value is new.

Theorem 6.7. *Let $G = (I, \{0,1\}, \lambda)$ be an XOR game with $n := |I|$. Let $\pi(x,y)$ be a prior distribution on I. Then*

$$\omega_{qc}(G,\pi) = \omega_q(G,\pi) = \frac{1}{2} + \frac{1}{2}\max_{P\in\mathcal{E}_{2n}} Tr(B_G P)$$

and

$$\omega_{qc}^s(G,\pi) = \omega_q^s(G,\pi) = \frac{1}{2} + \frac{1}{2}\max_{P\in\mathcal{E}_n} Tr(A_G^s P).$$

Proof. Suppose $f : I \times I \to \{0,1\}$ is a function satisfying $f(x,y) = a \oplus b$ if and only if $\lambda(x,y,a,b) = 1$ for all $a, b \in \{0,1\}$.

We first consider the claim concerning $\omega_{qc}(G,\pi)$. By Proposition 6.2, there exists $p \in C_q(n,2)$ such that $\omega_{qc}(G,\pi) = \omega(G,\pi,p)$ and $p_A(0|x) = p_B(0|y) = 1/2$ for every $x, y \in I$. Since $\lambda(x,y,a,b) = 1$ if and only if $a \oplus b = f(x,y)$, we have that

$$\omega_{qc}(G,\pi) = \sum_{x,y \in I, a\oplus b = f(x,y)} \pi(x,y) p(a,b|x,y).$$

By Theorem 6.3 this implies

$$\begin{aligned} \omega_{qc}(G,\pi) &= \sum_{x,y \in I, a\oplus b = f(x,y)} \frac{1}{4}\pi(x,y)(1 + (-1)^{a+b}\langle v_x, w_y\rangle) \\ &= \frac{1}{4} \sum_{x,y \in I, a\oplus b = f(x,y)} \pi(x,y) + \frac{1}{4}\sum_{x,y\in I} \pi(x,y)(-1)^{f(x,y)}\langle v_x, w_y\rangle \end{aligned}$$

where the v_x's and w_y's are real unit vectors. Since every expression of the form $p(a,b|x,y) = \frac{1}{4}[1 + (-1)^{a+b}\langle v_x, w_y\rangle]$ defines an element of $C_{qc}(n,2)$, we have

$$\omega_{qc}(G,\pi) = \frac{1}{4} \sum_{x,y \in I, a\oplus b = f(x,y)} \pi(x,y) + \frac{1}{4} \max_{v_x, w_y} \sum_{x,y\in I} \pi(x,y)(-1)^{f(x,y)}\langle v_x, w_y\rangle$$

where the maximization is over all sets of real unit vectors v_x and w_y. Since $\pi(x,y)$ is a probability distribution and $a \oplus b = f(x,y)$ for exactly two choices of pairs (a,b), we have that

$$\sum_{x,y \in I, a\oplus b = f(x,y)} \pi(x,y) = 2.$$

Also, notice that an $n \times n$ matrix has the form $(\langle v_x, w_y\rangle)_{x,y}$ for unit vectors v_x and w_y if and only if it is the upper right (or lower left) $n \times n$ corner of a matrix $P \in \mathcal{E}_{2n}$, since every element $P \in \mathcal{E}_{2n}$ has a Gram decomposition

$$P = (v_1 \dots v_n w_1 \dots w_n)^*(v_1 \dots v_n w_1 \dots w_n).$$

A computation yields the expression

$$\omega_{qc}(G,\pi) = \omega_q(G,\pi) = \frac{1}{2} + \frac{1}{2}\max_{P \in \mathcal{E}_{2n}} Tr(B_{\mathcal{G}}P).$$

To verify the claims concerning $\omega_{qc}^s(G,\pi)$, first note that by the above argument we have

$$\omega_{qc}^s(G,\pi) = \omega_q^s(G,\pi) = \frac{1}{2} + \frac{1}{2}\max_{P' \in \mathcal{E}'_{2n}} Tr(B_{\mathcal{G}}P').$$

where $\mathcal{E}'_{2n} \subseteq \mathcal{E}_{2n}$ is taken to be the set of $P \in \mathcal{E}_{2n}$ whose upper right $n \times n$ corner has the form $(\langle v_x, v_y\rangle)_{x,y}$ for a single set of real unit vectors

$\{v_1, \ldots, v_n\}$, by Theorem 6.4. Because of the form of $B_{\mathcal{G}}$, we may assume any $P' \in \mathcal{E}'_{2n}$ has the form

$$P' = \begin{pmatrix} P & P \\ P & P \end{pmatrix}, \quad P \in \mathcal{E}_n,$$

and a computation shows that $Tr(B_{\mathcal{G}}P') = Tr(A^s_G P)$. Thus

$$\omega^s_{qc}(G, \pi) = \omega^s_q(G, \pi) = \frac{1}{2} + \frac{1}{2} \max_{P \in \mathcal{E}_n} Tr(A^s_G P).$$

This proves the claims. □

7. Values of Products of Games

This section is also lifted directly from [11].

There is a great deal of research concerning products of games and especially their behaviour when one does many iterations of a fixed game.[14, 6, 2] Many of these results are false for synchronous values of games.

Given two games $G_i = (X_i, O_i, \lambda_i), i = 1, 2$ their product $G_1 \times G_2$ is the game with input set $X := X_1 \times X_2$, output set $O := O_1 \times O_2$ and rule function,

$$\lambda : X \times X \times O \times O \to \{0, 1\} = \mathbb{Z}_2,$$

given by

$$\lambda((x_1, x_2), (y_1, y_2), (a_1, a_2), (b_1, b_2)) = \lambda_1(x_1, y_1, a_1, b_1)\lambda_2(x_2, y_2, a_2, b_2),$$

where the product is in $\mathbb{Z}_2$. Thus, they win if and only if $\lambda_1(x_1, y_1, a_1, b_1) = 1$ and $\lambda_2(x_2, y_2, a_2, b_2) = 1$, that is if and only if they win both games. It is customary to write $\lambda = \lambda_1 \times \lambda_2$.

Given prior distributions $\pi_1 : X_1 \times X_1 \to [0, 1]$ and $\pi_2 : X_2 \times X_2 \to [0, 1]$ it is easy to see that by defining,

$$\pi : X \times X \to [0, 1], \;\; \pi((x_1, x_2), (y_1, y_2)) := \pi_1(x_1, y_1), \pi_2(x_2, y_2),$$

we obtain a distribution on $X \times X$, which is denoted by $\pi_1 \times \pi_2$.

If $\mathcal{G}_i = (G_i, \pi_i)$ denotes the game with distribution π_i then we set $\mathcal{G}_1 \times \mathcal{G}_2 = (G_1 \times G_2, \pi_1 \times \pi_2)$.

These definitions clearly extend to products of more than two games. Given a game with distribution $\mathcal{G} = (G, \pi)$ we let $\mathcal{G}^n = (G^n, \pi^n)$ denote the n-fold product of a game with itself.

Here are a few of the results that are known for the values of such games:

(1) (Supermultiplicativity) $\omega_t(\mathcal{G} \times \mathcal{H}) \geq \omega_t(\mathcal{G})\omega_t(\mathcal{H})$, and examples exist for which the inequality is strict,
(2) $\omega_t(\mathcal{G} \times \mathcal{H}) \leq min\{\omega_t(\mathcal{G}), \omega_t(\mathcal{H})\}$
(3) $G \times H$ has a perfect t-strategy $\iff$ G and H each have a perfect t-strategy for $t = loc, qa, qc$.
(4) if $\omega_{loc}(\mathcal{G}) < 1$, then $\omega_t(\mathcal{G}^n) \to 0$.

Thus, when the value is not 1, even though it is possible that $\omega_{loc}(\mathcal{G}^n) > \omega_{loc}(\mathcal{G})^n$, we still have that it tends to 0.

The analogues of (1) and (3) were shown to hold for synchronous values in [20], where an example is also given to show that the inequality can be strict.

The example below shows that (2) and (4) can fail for synchronous values.

Example 7.1. Let $\mathcal{G} = (G, \pi)$ be the game where Alice's and Bob's question and answer sets are $\{0, 1\}$ and let the distribution π be given by $\pi_{0,1} = \pi_{1,1} = 1/2$. The players win if their answer pair is $(1, 1)$ when asked question pair $(0, 1)$. They also win if their answer pair is $(0, 1)$ when they receive question pair $(1, 1)$. They lose in all other cases. Note that Bob receives 1 with probability 1 while Alice receives $0, 1$ with equal probability.

This game has a perfect non-synchronous strategy, namely, for Bob to always return 1 and for Alice given input $x \in \mathbb{Z}_2$ to always return $x + 1$. Thus,

$$\omega_{loc}(\mathcal{G}) = \omega_{qc}(\mathcal{G}) = 1,$$

and consequently,

$$\omega_{loc}(\mathcal{G}^n) = \omega_{qc}(\mathcal{G}^n) = 1.$$

Theorem 7.2. *Let $\mathcal{G} = (G, \pi)$ be the game with distribution of Example 7.1. Then*

$$\omega^s_{loc}(\mathcal{G}^n) = \omega^s_{qc}(\mathcal{G}^n) = 1 - \frac{1}{2^n}.$$

Proof. The synchronous value of this game is at most $1/2$, since on question $(1, 1)$ a synchronous strategy will require them to return the same answer and lose. On the other hand, the deterministic strategy of Alice and Bob always returning 1 has a value of $1/2$. Hence, $\omega^s_{loc}(G) = \omega^s_q(G) = \frac{1}{2}$. In terms of traces and projections, this is given by setting $E_{0,1} = E_{1,1} = I$ and $E_{0,0} = E_{1,0} = 0$.

Now for the n-fold parallel repetition the questions are pairs $x, y \in \{0, 1\}^n$ and the answers are pairs $a, b \in \{0, 1\}^n$. But $\pi^n(x, y) = 0$ unless $y = (1, ..., 1) := 1^n$, while $\pi(x, 1^n) = \frac{1}{2^n}, \ \forall x \in \{0, 1\}^n$.

The only question pair where the synchronous restriction can be enforced is therefore $(1^n, 1^n)$, and on this question any synchronous strategy loses as before. Thus, $\omega^s_{qc}(\mathcal{G}^n) \leq 1 - \frac{1}{2^n}$.

On the other hand, consider the deterministic strategy where when the input string is 1^n they return 1^n but for every other input string $x \neq 1^n$, they return the output string $\overline{x} = x + 1^n$, where addition is in the vector space $\mathbb{Z}_2^n$, i.e., each bit of x is flipped. For every string $x \neq 1^n$ that Alice receives this strategy wins. Hence, $\omega^s_{loc}(\mathcal{G}^n) \geq 1 - \frac{1}{2^n}$. Therefore the synchronous value of the parallel repeated game is $\omega^s_{loc}(G^n) = \omega^s_{qc}(G^n) = 1 - \frac{1}{2^n}$.

Alternatively, this is the strategy that corresponds to choosing PVM's,

$$E_{1^n, 1^n} = E_{x, \overline{x}} = I, \ \forall x \neq 1^n,$$

and all other projections equal to 0. □

Thus, not only does the synchronous value not tend to 0, but it is monotonically increasing. Also, we have that

$$\omega_t^s(\mathcal{G}^2) > \min\{\omega_t(\mathcal{G}), \omega_t^s(\mathcal{G})\},$$

so that this example violates the synchronous analogues of properties (2) and (4).

Two objections can be raised to this example. The game itself is not synchronous and the distribution is not symmetric. It is natural to wonder if this pathology persists even when restricting attention to this smaller family of synchronous games with symmetric prior densities. This is formalized in the following problems.

Problem 7.3. *If $\mathcal{G}_i = (G_i, \pi_i), i = 1, 2$ are symmetric synchronous games with symmetric densities, then Is $\omega_t^s(\mathcal{G}_1 \times \mathcal{G}_2) \leq \min(\omega_t^s(\mathcal{G}_1), \omega_t^s(\mathcal{G}_2))$?*

Problem 7.4. *If $\mathcal{G}$ is a symmetric, synchronous game with symmetric distribution, can $\omega_t^s(\mathcal{G}^n)$ be monotone increasing?*

We next return our attention to XOR games.

First note that the product of two XOR games is not an XOR game. In fact the product is not even a game with binary answers. Our first step is to recall an operation on XOR games, studied in [4], that unlike the product, produces an XOR game. The **XOR of XOR games** G_1 and G_2 with densities π_1, π_2 and rule functions f_1 and f_2, denoted by $G_1 \oplus G_2$, is the XOR game $(I_1 \times I_2, \{0, 1\}, \lambda)$ with distribution $\pi_1 \times \pi_2$ and rule function λ defined so that $\lambda((x_1, x_2), (y_1, y_2), a, b) = 1$ iff $a+b = f_1(x_1, y_1)+f_2(x_2, y_2)$ in $\mathbb{Z}_2$. The XOR of more than two games is defined inductively. The following result shows why this is an interesting operation on XOR games.

Proposition 7.5. *Let $\mathcal{G}_i = (I_i, \{0, 1\}, \lambda_i, \pi_i), i = 1, 2$ be XOR games with densities and cost matrices $A_{\mathcal{G}_i}, i = 1, 2$. Then the cost matrix of their direct sum satisfies*

$$A_{\mathcal{G}_1 \oplus \mathcal{G}_2} = A_{\mathcal{G}_1} \otimes A_{\mathcal{G}_2}.$$

The **bias** of a game with distribution is defined by the formulas

$$\epsilon_t(\mathcal{G}) = 2\omega_t(\mathcal{G}) - 1, \; t = loc, q, qc,$$

and corresponds to the probability of winning minus the probability of losing. Similarly, we have the **synchronous bias**,

$$\epsilon_t^s(\mathcal{G}) = 2\omega_t^s(\mathcal{G}) - 1, \; t = loc, q, qc.$$

In [4, Theorem 1] it was proven that the quantum bias of XOR games is multiplicative for the direct sum operations, i.e.,

$$\epsilon_q(\mathcal{G}_1 \oplus \mathcal{G}_2) = \epsilon_q(\mathcal{G}_1)\epsilon_q(\mathcal{G}_2).$$

.

In what follows we show that this fails for the synchronous bias, even for a family of games that is very well behaved.

Definition 7.6. An XOR game with distribution π will be called a **synchronous XOR game**, provided that the game is synchronous, i.e., $f(x,x) = 0$, symmetric, $f(x,y) = f(y,x)$ and the distribution is symmetric, $\pi(x,y) = \pi(y,x)$.

.

Note that when $\mathcal{G}$ is a synchronous XOR game, we have that the cost matrix $A_{\mathcal{G}} = ((-1)^{f(x,y)}\pi(x,y)) = A_{\mathcal{G}}^T$ and hence,

$$A_{\mathcal{G}}^s = A_{\mathcal{G}}.$$

In what follows we first show that the perfect parallel repetition does not hold for the synchronous bias of synchronous XOR games. We then identify a subclass of XOR games for which the synchronous value satisfies the perfect parallel repetition.

Restating Theorem 6.7 in terms of biases yields:

Theorem 7.7. *Let $G = (I, \{0,1\}, \lambda)$ be an XOR game with $n := |I|$, and suppose $f : I \times I \to \{0,1\}$ is a function satisfying $f(x,y) = a \oplus b$ for all $a, b \in \{0,1\}$. Let $\pi(x,y)$ be a prior distribution on I. Then for $\mathcal{G} = (G, \pi)$,*

$$\epsilon_{qc}(\mathcal{G}) = \epsilon_q(\mathcal{G}) = \max_{P \in \mathcal{E}_{2n}} Tr(B_{\mathcal{G}} P)$$

and

$$\epsilon_{qc}^s(\mathcal{G}) = \epsilon_q^s(\mathcal{G}) = \max_{P \in \mathcal{E}_n} Tr(A_{\mathcal{G}}^s P).$$

Fix the question set to be $I = \{1, \dots, m\}$ and we can equivalently write the above optimization problem for the bias of a synchronous XOR game as the primal-dual semidefinite programs

$$(\mathcal{P}) \quad \text{maximize:} \quad \langle A|P\rangle \qquad\qquad (\mathcal{D}) \quad \text{minimize:} \quad \sum_{k=1}^m y_k$$
$$\text{subject to:} \quad diag(P) = 1, \qquad\qquad \text{subject to:} \quad Diag(y) - A \succeq 0,$$
$$P \geq 0,$$

where the inner product is the trace inner product,

$$A := A_{\mathcal{G}}^s = 1/2(\pi(x,y)(-1)^{f(x,y)}) + 1/2(\pi(x,y)(-1)^{f(x,y)})^T,$$

and $diag$ is the function that zeros out nondiagonal entries of a matrix, and $Diag$ of a vector is the matrix where the diagonal entries are the vector entries and nondiagonal entries are zero. This primal-dual satisfies the Slater condition [29] and therefore their optimal values are attained and are equal. In fact by complementary slackness if (P^*, y^*) is an optimal solution pair for primal and dual then it holds that $P^*(Diag(y^*) - A) = 0$. Now if y' is any other optimal dual solution, it holds that $P^* Diag(y^* - y') = 0$. Since the diagonal entries of P are 1, this implies that $y' = y^*$. Therefore we get the following lemma

Lemma 7.8. *The dual problem $(\mathcal{D})$ has a unique optimal solution.*

In the next theorem, we show that the bias of an XOR game for which $Diag(y^*) \geq A \geq -Diag(y^*)$, where y^* is the unique dual optimal solution, are multiplicative. That is for any two XOR games with this property, we have $\epsilon_q^s(G_1 \oplus G_2) = \epsilon_q^s(G_1)\epsilon_q^s(G_2)$. This in particular includes all XOR games for which the game matrix is positive semidefinite. This is not true for all XOR games as is shown by the next example.

Example 7.9. Let $\mathcal{G}$ be the synchronous XOR game with cost matrix

$$A = \begin{bmatrix} \frac{1}{21} & -\frac{3}{21} & -\frac{3}{21} \\ -\frac{3}{21} & \frac{1}{21} & -\frac{3}{21} \\ -\frac{3}{21} & -\frac{3}{21} & \frac{1}{21} \end{bmatrix}.$$

The pair $P^* = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$ and $y^* = \begin{bmatrix} \frac{4}{21} \\ \frac{4}{21} \\ \frac{4}{21} \end{bmatrix}$ are easily seen to be feasible solutions of the primal and dual SDPs and they achieve the same value $\frac{4}{7}$ in the primal and dual problems, respectively. Therefore they are optimal solutions and the optimal value and hence the synchronous quantum bias of this game is

$$\epsilon_q^s(\mathcal{G}) = \frac{4}{7}.$$

Now the cost matrix for the game $\mathcal{G}' = \mathcal{G} \oplus \mathcal{G}$ is $A \otimes A$. Therefore the primal-dual problem for $\mathcal{G}'$ is

$$\begin{aligned} (\mathcal{P}) \quad \text{maximize:} \quad & \langle A \otimes A | W \rangle \\ \text{subject to:} \quad & diag(W) = 1, \\ & W \succeq 0, \end{aligned} \qquad \begin{aligned} (\mathcal{D}) \quad \text{minimize:} \quad & \sum_{k=1}^{9} u_k \\ \text{subject to:} \quad & Diag(u) - A \otimes A \succeq 0. \end{aligned}$$

Now from a similar argument like above the pair $W^* = ee^*$ where $e \in \mathbb{C}^9$ is the all-one vector and $u = (\frac{5}{21})^2 e$ are optimal solutions for the primal and dual respectively and the optimal value is $(\frac{5}{7})^2$. So we have that

$$\epsilon_q^s(\mathcal{G} \oplus \mathcal{G}) = (\frac{5}{7})^2 > (\frac{4}{7})^2 = \epsilon_q^s(\mathcal{G})^2.$$

Note that the unique optimal solution y^* for the dual problem of G does not satisfy the condition

$$Diag(y^*) \geq A \geq -Diag(y^*)$$

because the eigenvalues of A are $4/21, 4/21, -5/21$.

Definition 7.10. We call a synchronous XOR game $\mathcal{G}$ and symmetrized cost matrix $A := A_{\mathcal{G}}^s$ **balanced**, if the unique optimal dual solution y^* satisfies

$$Diag(y^*) \geq A \geq -Diag(y^*).$$

Suppose that $\mathcal{G}$ is a balanced game and y^* is its unique dual optimal solution. Note that if $y^*(i) \le 0$ for some question i, then the inequalities above imply that $y^*(i) = A(i,i) = 0$. Then again since $A + Diag(y^*)$ is positive semidefinite (and its ith diagonal element is 0), it must be that the ith column and row of A are all zeros. Therefore it is true that $\pi(i,j) = \pi(j,i) = 0$ for all questions j. Therefore question i is irrelevant and can be removed from the question set of the original game. Thus without loss of generality, we can assume that $y^* > 0$.

Proposition 7.11. *Any XOR $\mathcal{G}$ game for which $A^s_{\mathcal{G}} \ge 0$ is balanced.*

Theorem 7.12. *If $\mathcal{G}_i, i = 1, 2$ are balanced XOR games, then*

$$\epsilon^s_q(\mathcal{G}_1 \oplus \mathcal{G}_2) = \epsilon^s_q(\mathcal{G}_1)\epsilon^s_q(\mathcal{G}_2)$$

and $\mathcal{G}_1 \oplus \mathcal{G}_2$ is balanced.

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