# Embezzlement of Entanglement

Vern Paulsen

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They referred to this as *embezzlement of entanglement.*
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Thus we have a "task" that can be carried out to an arbitrary degree of accuracy in finite dimensions, but even as we let the dimensions become infinite, we still cannot carry it out exactly.

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Note that

$$(U_A \otimes I_{\mathcal{R}_B} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes I_{\mathcal{R}_A} \otimes U_B) = U_A \otimes U_B = (I_{\mathcal{H}_A} \otimes I_{\mathcal{R}_A} \otimes U_B)(U_A \otimes I_{\mathcal{R}_B} \otimes I_{\mathcal{H}_B}).$$

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Instead, we only ask for a resource space $\mathcal{R}$, and unitaries, $U_A$ on $\mathcal{H}_A \otimes \mathcal{R}$ and $U_B$ on $\mathcal{R} \otimes \mathcal{H}_B$ such that $(U_A \otimes id_B)$ commutes with $(id_A \otimes U_B)$ on $\mathcal{H}_A \otimes \mathcal{R} \otimes \mathcal{H}_B$.

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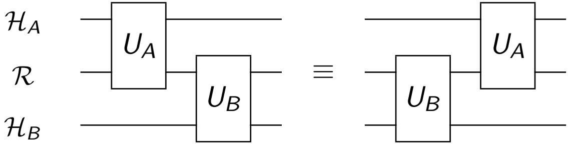

## Theorem (Cleve-Liu-P, Harris-P)

*Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional. Given any unit vector $\phi = \sum_{i,j} \alpha_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exists a Hilbert space $\mathcal{R}$, a unit vector $\psi \in \mathcal{R}$, unitaries*

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Briefly, catalytic production of entanglement is possible in the commuting operator model.

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### Lemma

*$(U_A \otimes id_B)$ commutes with $(id_A \otimes U_B)$ if and only if $U_{i,j} V_{k,l} = V_{k,l} U_{i,j}$ and $U_{i,j}^* V_{k,l} = V_{k,l} U_{i,j}^*$ for all $i, j, k, l$.*

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Thus, we see that having commuting operator frameworks as above is exactly the same as having operator matrices $U_A = (U_{i,j})$ and $U_B = (V_{k,l})$ that yield unitaries and whose entries pairwise *-commute.

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Thus, a representation of $U_{nc}(n) \otimes_{max} U_{nc}(m)$ corresponds to operators $U_{i,j}, V_{k,l}$ where the $U_{i,j}$'s *-commute with the $V_{k,l}$'s such that $(U_{i,j})$ and $(V_{k,l})$ are unitary operator matrices.

### Theorem (Cleve-Liu-P, Harris-P)

*Perfect embezzlement of a state* $\phi = \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_{i,k} |i\rangle \otimes |k\rangle$ *is possible in a commuting operator framework if and only if there is a state* $s$ *on* $U_{nc}(n) \otimes_{max} U_{nc}(m)$ *satisfying* $s(u_{i,1} \otimes v_{k,1}) = \alpha_{i,k}$*.*

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The approximate embezzlement results yield states on $U_{nc}(n) \otimes_{min} U_{nc}(m)$ that converge to a state on $U_{nc}(n) \otimes_{min} U_{nc}(m)$ satisfying the above equations, and hence the desired state on $U_{nc}(n) \otimes_{max} U_{nc}(m)$.

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The occurrence of min and max tensors in different places lead me to wonder what is their relationship? Maybe they are the same?

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**Theorem (Harris)**

*The following are equivalent.*

1. *Connes' Embedding conjecture is true.*
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The equivalence of the first three, is the analogue of Kirchberg's theorem relating Connes to tensor products of free group C*-algebras. The equivalence of the first and last is the analogue of the results of Junge, Navascues, Palazuelas, Perez-Garcia, Scholz, Werner and separately, Ozawa, relating CEP to Tsirelson's problems.

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The set $B_{qc}(n,m)$ is defined similarly except we replace the tensor product of two spaces by a single space and instead demand that the $U_{i,j}$'s *-commute with the $V_{k,l}$'s.

Suppose that we are given $U_{i,j} \in M_p$, $1 \le i,j \le n$ such that $U = (U_{i,j}) \in M_n(M_p)$ is unitary and $V_{k,l} \in M_q$, $1 \le k,l \le m$ such that $V = (V_{k,l}) \in M_m(M_p)$ is unitary. Let $\psi \in \mathbb{C}^p \otimes \mathbb{C}^q$ with $||\psi|| = 1\}$, and set $x_{i,j,k,l} = \langle \psi | (U_{i,j} \otimes V_{k,l}) \psi \rangle$.
We let $B_q(n,m) \subseteq M_n \otimes M_m$ denote the set of all matrices $X = (x_{i,j,k,l})$ obtained in this manner.
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Here are some of the things that we know/don't know about these sets.

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Harris showed that $B_q(2, 2)$ not closed implies that $C_q(8, 8)$ not closed.

Next we give an operational meaning to these sets.

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- *CEP is true iff* $bias_q(G) = bias_{qc}(G), \forall G$.

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*Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional. If $\psi = \sum_{i,j} \beta_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and its highest Schmidt coefficient satisfies $\lambda_1 \leq \sqrt{\frac{2}{3}}$,*

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$$\|U_A \otimes U_B(|0\rangle \otimes \psi \otimes |0\rangle) - \sum_{i,j} \beta_{i,j} |i\rangle \otimes \psi \otimes |j\rangle\| \geq \frac{2}{3}(3 - 2\sqrt{2})$$

*and this bound is independent of the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$.*

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$$(U_A \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes U_B)(|0\rangle \otimes \psi \otimes |0\rangle) = \sum_{i,j} \beta_{i,j} |i\rangle \otimes \psi \otimes |j\rangle \simeq \psi \otimes \psi.$$

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# References

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Thanks!