Rigorous results on many-body localization in quantum spin chains

Günter Stolz
Centre for the Mathematics of Quantum Theory
Copenhagen, May 2017

Part 1:

Basic theory of the XY Spin Chain

Going back to:

Lieb-Schultz-Mattis 1961 (Katsura 1962, Pfeuty 1970, Barouch and McCoy 1971)

L interacting $\frac{1}{2}$-spins:

$$\mathcal{H}_L = \mathbb{C}^2_1 \otimes \ldots \otimes \mathbb{C}^2_L, \quad \dim \mathcal{H} = 2^L$$

Canonical (up/down-spin) product basis:

$$e_0 = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1 = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(downspins = particles)}$$

$$e_\alpha = e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_L}, \quad \alpha \in \{0,1\}^L$$

Local observables: For $A \in \mathbb{C}^{2\times 2}$ let

$$A_j = I \otimes \ldots \otimes A \otimes \ldots \otimes I \quad \text{(acting non-trivially on } j\text{-th spin)}$$

Pauli matrices:

$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin raising and spin lowering operators:

$$a := \frac{1}{2}(\sigma^X + i\sigma^Y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a^* := \frac{1}{2}(\sigma^X - i\sigma^Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$aa^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a^*a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: \mathcal{N}$$

Isotropic XY spin chain in transversal field:

$$\begin{aligned} H &= H_{XY} = -\sum_{j=1}^{L-1} \mu_j (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) - \sum_{j=1}^{L} \omega_j \sigma_j^Z \\ &= -2 \sum_{j=1}^{L-1} \mu_j (a_j^* a_{j+1} + a_{j+1}^* a_j) - \sum_{j=1}^{n} \omega_j (I - 2 a_j^* a_j) \end{aligned}$$

Variable coefficients: $\omega_j \in \mathbb{R}$, $\mu_j \in \mathbb{R} \setminus \{0\}$

Jordan-Wigner transform:

$$c_1 := a_1, \quad c_j := \sigma_1^Z \dots \sigma_{j-1}^Z a_j, \quad j = 2, \dots, n$$

Canonical anti-commutation relations (CAR):

$$\{c_j, c_k^*\} = \delta_{jk} I, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0$$

Fermionic represenation of H:

$$\begin{aligned} H &= -2\sum_{j=1}^{L-1} \mu_j (c_j^* c_{j+1} + c_{j+1}^* c_j) - \sum_{j=1}^{L} \omega_j (I - 2c_j^* c_j) \\ &= 2c^* \, Mc + E_0 I \end{aligned}$$

Here $E_0 := -\sum_j \omega_j$, $c^* := (c_1^*, \ldots, c_L^*)$,

$$c := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad M := \begin{pmatrix} \omega_1 & -\mu_1 & & \\ -\mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & -\mu_{L-1} \\ & & -\mu_{L-1} & \omega_L \end{pmatrix}$$

M is the **effective one-particle Hamiltonian** of the XY chain, acting on an L-dimensional space.

Bogolubov transformation (special):

M real symmetric, so there exists orthogonal U such that

$$UMU^t = \Lambda = \text{diag}\,(\lambda_j)$$

Let

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_L \end{pmatrix} := Uc$$

$\Longrightarrow \{b_j\}_{j=1}^{L}$ satisfy CAR and

$$H = 2\sum_{j=1}^{L} \lambda_j b_j^* b_j + E_0 I = 2b^* \Lambda b + E_0 I$$

Free Fermion system!

Properties of the Fermionic operators $\{b_j\}_{j=1}^L$:

- $b_j^* b_j$, $j = 1, \ldots, L$, pairwise commuting orthogonal projections
- $N = \bigcap_{j=1}^L \ker(b_j^* b_j) = \bigcap_{j=1}^L \ker(b_j)$ one-dimensional.
  Pick normalized $\Omega \in N$ ("vacuum state").
- $\psi_\alpha := (b_1^*)^{\alpha_1} \ldots (b_L^*)^{\alpha_L} \Omega$, $\alpha \in \{0,1\}^L$ form ONB of $\mathcal{H}$.
- All ψ_α are eigenvectors of each $b_j^* b_j$:

$$b_j^* b_j \psi_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 0 \\ 1 & \text{if } \alpha_j = 1 \end{cases}$$

Eigenvectors and eigenvalues of H:

All ψ_α are also eigenvectors of H:

$$H\psi_\alpha = \left(2\sum_{j:\alpha_j=1} \lambda_j + E_0\right)\psi_\alpha$$

$$\sigma(H) = \left\{2\sum_{j:\alpha_j=1} \lambda_j + E_0 : \ \alpha \in \{0,1\}^L\right\}$$

Ground state energy:

$$2\sum_{j=1}^{L} \min\{0,\lambda_j\} + E_0$$

$$\text{non-degenerate} \quad \Longleftrightarrow \quad \lambda_j \neq 0 \text{ for all } j$$

Finding eigenvalues and eigenvectors of H has been reduced to finding eigenvalues and eigenvectors of

$$M = \begin{pmatrix} \omega_1 & -\mu_1 & & \\ -\mu_1 & \omega_2 & \ddots & \\ & \ddots & \ddots & -\mu_{L-1} \\ & & -\mu_{L-1} & \omega_L \end{pmatrix}$$

Lieb, Schultz, Mattis (and others in 1960s):

$$\omega_j = C_1, \quad \mu_j = C_2 \quad \Longrightarrow \quad \text{Exactly solvable!}$$

In general: Dimension of Hilbert space reduced from 2^L to L.
Hope: Qualitative properties of M imply qualitative properties of H.

Exercise 1 (Dynamics of c_j):

Denote the Heisenberg dynamics of $A \in B(\mathcal{H}_L)$ by

$$\tau_t(A) := e^{itH} A e^{-itH}$$

Show that

$$\tau_t(c_j) = \sum_{\ell} \left(e^{-2iMt} \right)_{j\ell} c_\ell$$

Thus: One-particle dynamics e^{-2iMt} determines many-body dynamics $\tau_t(c_j)$.

Problem: Jordan-Wigner is non-local. How to “undo” Jordan-Wigner to get dynamics of local operators such as a_j, a_j^*?

Solution to Exercise 1:

Lemma

$$\tau_t(b_k) = e^{-2it\lambda_k} b_k, \quad \tau_t(b_k^*) = e^{2it\lambda_k} b_k^*, \quad k = 1, \ldots, L$$

Proof.

$$\begin{aligned} \frac{d}{dt}\tau_t(b_k) &= -i\tau_t([b_k, H]) \\ &= -2i\sum_j \lambda_j \tau_t([b_k, b_\ell^* b_\ell]) \\ &= -2i\lambda_k \tau_t([b_k, b_k^* b_k]) = -2i\lambda_k \tau_t(b_k) \end{aligned}$$

Also: $\tau_0(b_k) = b_k$. Unique solution: $\tau_t(b_k) = e^{-2it\lambda_k} b_k$

□

Solution (cont.):

(i) $b = Uc$
(ii) Lemma: $\tau_t(b) = e^{-2it\Lambda}b$
(iii) $UMU^t = \Lambda \Longrightarrow e^{-iMt} = U^t e^{-it\Lambda}U$

Thus:

$$\begin{aligned}
\tau_t(c) &= e^{itH}ce^{-itH} \\
&= e^{itH}U^t be^{-itH} \\
&= U^t e^{itH}be^{-itH} \\
&= U^t e^{-2it\Lambda}b \\
&= U^t e^{-2it\Lambda}Uc \\
&= e^{-2iMt}c \qquad \text{q.e.d.}
\end{aligned}$$

Particle number conservation and second quantization:

Isotropic XY preserves number of down spins: H leaves

$$\mathcal{H}_L^{(N)} := \text{span}\{e_\alpha : \sum \alpha_j = N\}$$

invariant for all $N = 0, \ldots, L$. Let $H_N := H|_{\mathcal{H}_L^{(N)}}$.

$$H = \bigoplus_{N=0}^{L} H_N$$

Second Quantization:

In fact:

$$\begin{array}{rcl} H_0 & = & E_0 \quad \text{on span}\{|\uparrow\uparrow\ldots\uparrow\uparrow\rangle\} \text{ (vacuum)} \\ H_1 & \cong & 2M + E_0 \quad \text{on } \mathbb{C}^L \\ H_N & \cong & 2M^{\wedge N} + E_0 \quad \text{on } \bigwedge^N(\mathbb{C}^L) \end{array}$$

Thus

$$H = 2c^* Mc + E_0 \cong 2d\Gamma_a(M) + E_0,$$

where $d\Gamma_a(M)$ is the restriction of the 2nd quantization of M on the antisymmetric Fock space

$$\mathcal{F}_a(\mathbb{C}^L) \cong \mathbb{C}^{2^L}.$$

Part 2:

MBL Properties of the Disordered XY Chain

Survey of some of the results from:

Hamza/Sims/St. 2012
Klein/Perez 1992, Sims/Warzel 2016
Pastur-Slavin 2014, Abdul-Rahman/St. 2015
Abdul-Rahman/Nachtergaele/Sims/St. 2016 (Survey 2017)

XY Chain in Random Field:

$$H = -\sum_{j=1}^{L-1}(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) - \sum_{j=1}^{L} \omega_j \sigma_j^Z \quad \text{on } \mathcal{H}_L = \bigoplus_{j=1}^{L} \mathbb{C}_j^2$$

Assume: $(\omega_j)_{j=1}^{\infty}$ i.i.d. random variables, with distribution $d\mu(\omega_j) = \rho(\omega_j)\, d\omega_j$, where ρ is bounded and compactly supported.

Effective Hamiltonian:

$$M = \begin{pmatrix} \omega_1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \omega_L \end{pmatrix}$$

1D Anderson Model!

Known strong form of Anderson Localization:

Eigencorrelator Localization:

$$\mathbb{E}\left(\sup_{|g|\leq 1} |(g(M))_{jk}|\right) \leq Ce^{-\mu|j-k|}$$

uniformly in L. (e.g. Aizenman-Warzel book)

In particular: **Dynamical Localization:**

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} |(e^{-itM})_{jk}|\right) \leq Ce^{-\mu|j-k|}$$

This implies various forms of MBL-type properties for the XY chain:

Zero-velocity Lieb-Robinson bound

$A, B \in \mathbb{C}^{2\times 2}$
$A_j = I \otimes \ldots \otimes A \otimes \ldots \otimes I$ (in j-th position), $B_k = \ldots$
$\tau_t(A_j) = e^{itH} A_j e^{-itH}$

Theorem 1 (Hamza/Sims/St. 2012)
There exist $C < \infty$ and $\mu > 0$ such that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau_t(A_j), B_k]\|\right) \leq C\|A\|\|B\|e^{-\mu|j-k|}$$

for all L, $1 \leq j, k \leq L$, $A, B \in \mathbb{C}^{2\times 2}$.

Note: Requires averaging over disorder $\mathbb{E}(\cdot)$.

Compare: Lieb-Robinson 1972 (and others more recently):

For a quite general class of quantum spin systems (with bounded coefficients and bounded interaction range) it holds that

$$\|[\tau_t(A_j), B_k]\| \leq C\|A\|\|B\|e^{-\mu(|j-k|-v|t|)}$$

$$v < \infty \quad \text{group velocity}$$

Deterministic result!

The **Proof of Theorem 1** requires little more than the result of Exercise 1 above and summing up two geometric series, see Hamza/Sims/St. 2012.

Exponential Decay of Correlations ("Exponential Clustering"):

Theorem 2 (Sims/Warzel 2016)
There exist $C < \infty$ and $\mu > 0$ such that

$$\mathbb{E}\left(\sup_{\psi,t} |\langle\psi, \tau_t(A_j)B_k\psi\rangle - \langle\psi, A_j\psi\rangle\langle\psi, B_k\psi\rangle|\right) \leq C\|A\|\|B\|e^{-\mu|j-k|}$$

for all L, $1 \leq j, k \leq L$, and all $A, B \in \mathbb{C}^{2\times 2}$.
Here the supremum is taken over **all** normalized eigenfunctions ψ of H and **all** $t \in \mathbb{R}$.

Notes: (i) Same for thermal states $\rho_\beta = e^{-\beta H}/\mathrm{Tr} e^{-\beta H}$, with expectations defined as $\langle A\rangle_{\rho_\beta} = \mathrm{Tr}\rho_\beta A$.
(ii) Earlier related work (ground state): Klein/Perez 1992

Area Law for the Entanglement Entropy:

Bipartite decomposition:

$$\mathcal{H}_L = \mathcal{H}_A \otimes \mathcal{H}_B, \quad \mathcal{H}_A = \bigotimes_{j=1}^{\ell} \mathbb{C}_j^2, \quad \mathcal{H}_B = \bigotimes_{j=\ell+1}^{L} \mathbb{C}_j^2$$

ψ normalized eigenstate of H, $\rho_\psi = |\psi\rangle\langle\psi|$, reduced state:

$$\rho_\psi^A = \mathrm{Tr}_B\, \rho_\psi$$

Bipartite entanglement entropy:

$$\mathcal{E}(\rho_\psi) := \mathcal{S}(\rho_\psi^A) := -\mathrm{Tr}\, \rho_\psi^A \log \rho_\psi^A$$

Uniform Area Law:

Theorem 3: (Abdul-Rahman/St. 2015)
There exists $C < \infty$ such that

$$\mathbb{E}\left(\sup_{\psi} \mathcal{E}(\rho_\psi)\right) \leq C$$

for all L and all $1 \leq \ell < L$. Here the supremum is taken over **all** normalized eigenstates ψ of H.

Notes: (i) Method due to Pastur/Slavin 2014, who proved the area law for the ground state of a disordered d-dimensional quasi-free Fermion system.
(ii) No logarithmic correction in ℓ.
(iii) Open problem: Analogue for thermal states? (Problem: $\mathcal{E}$ is not a good entanglement measure for mixed states.

Entanglement Dynamics:

Let

- H_A, H_B restrictions of H to A and B
- ψ_A, ψ_B normalized eigenstates of H_A, H_B, $\rho_A = |\psi_A\rangle\langle\psi_A|$, $\rho_B = |\psi_B\rangle\langle\psi_B|$
- $\rho = \rho_A \otimes \rho_B$ (i.e. $\mathcal{E}(\rho) = 0$)
- $\rho_t = e^{-itH}\rho e^{itH}$ Schrödinger dynamics ("quantum quench")

Theorem 4: (Abdul-Rahman/Nachtergaele/Sims/St. 2016)
There exists $C < \infty$ such that

$$\mathbb{E}\left(\sup_{t,\psi_A,\psi_B} \mathcal{E}(\rho_t) \right) \leq C$$

for all ℓ and L.

Main tool in proofs of Theorems 2 to 4:

Quasifree States and their Correlation Matrices

Fact: Eigenstates $\rho = \rho_\alpha$, $\alpha \in \{0,1\}^L$, and thermal states $\rho = \rho_\beta$, $0 < \beta < \infty$, of a quasifree Fermion system $c^* M c$ are quasifree (i.e. expectations of arbitrary products of the c_j and c_j^* can be calculated by Wick's Rule.

Also: The reduced state ρ^A of a quasifree state ρ is again quasifree.

Thus ρ is uniquely determined by its correlation matrix

$$\Gamma_\rho = (\langle c_j c_k^* \rangle_\rho)_{j,k=1}^{L}$$

and ρ^A by the restricted correlation matrix

$$\Gamma_\rho^A = (\langle c_j c_k^* \rangle_\rho)_{j,k=1}^{\ell}$$

In particular, the proof of Theorem 2 is based on (Vidal/Latorre/Rico/Kitaev 2003):

$$\mathcal{S}(\rho) = -\mathrm{Tr}\,\rho \log \rho = \mathrm{tr}\, h(\Gamma_\rho)$$

where $h(x) = -x \log x - (1-x) \log(1-x)$. Same for $\mathcal{S}(\rho^A)$.

If $\sigma(M) = \{\lambda_j : j = 1, \dots, L\}$ is simple, then $\Gamma_{\rho_\alpha} = \chi_{\Delta_\alpha}(M)$, where

$$\Delta_\alpha := \{\lambda_j : \alpha_j = 0\}$$

The proof of Theorem 2 uses that, by Anderson localization of M,

$$\mathbb{E}\left(\sup_\alpha |(\chi_{\Delta_\alpha}(M))_{jk}|\right) \le Ce^{-\mu|j-k|}$$

See Pastur/Slavin 2014 and Abdul-Rahman/St. 2015 for more details.

Exercise 2: Correlation matrix of a free Fermion system

Show that $\Gamma_{\rho_\alpha} = \chi_{\Delta_\alpha}(M)$ if $\sigma(M)$ is simple.

Recall:

$$UMU^t = \operatorname{diag}(\lambda_j)$$

$$b = Uc$$

$$\Gamma_{\rho_\alpha}(j,k) = \langle \psi_\alpha, c_j c_k^* \psi_\alpha \rangle$$

$$\psi_\alpha = (b_1^*)^{\alpha_1} \dots (b_L^*)^{\alpha_L} \Omega$$

Solution to Exercise 2:

Step 1:

$$\begin{aligned}\langle \psi_\alpha, b_j b_k^* \psi_\alpha \rangle &= \langle b_j^* \psi_\alpha, b_k^* \psi_\alpha \rangle \\ &= \begin{cases} 0, & \text{if } j \neq k \text{ or } \alpha_j = 1 \text{ or } \alpha_k = 1, \\ 1, & \text{if } j = k \text{ and } \alpha_j = 0. \end{cases}\end{aligned}$$

Step 2: ONB f_j with $Mf_j = \lambda_j f_j$, thus $f_j(k) = U(j,k)$.

$$c_j = \sum_\ell f_\ell(j) b_\ell, \;\; c_k^* = \sum_r f_r(k) b_r^*$$

Thus

$$\begin{aligned}
\Gamma_{\rho_\alpha}(j,k) &= \langle \psi_\alpha, c_j c_k^* \psi_\alpha \rangle \\
&= \sum_{\ell,r} f_\ell(j) f_\ell(k) \langle \psi_\alpha, b_\ell b_r^* \psi_\alpha \rangle \\
&= \sum_{\ell} f_\ell(j) f_\ell(k) \langle \psi_\alpha, b_\ell b_\ell^* \psi_\alpha \rangle \\
&= \sum_{\ell:\alpha_\ell=0} f_\ell(j) f_\ell(k) \\
&= \sum_{\ell:\alpha_\ell=0} \langle \delta_j, f_\ell \rangle \langle f_\ell, \delta_k \rangle \\
&= \langle \delta_j, \chi_{\Delta_\alpha}(M) \delta_k \rangle \quad \text{(simplicity)}
\end{aligned}$$

Without providing details we mention that most of the above can be extended to the anisotropic XY chain in random field:

$$\begin{aligned} H_\gamma &= -\sum_{j=1}^{L-1}((1+\gamma)\sigma_j^X\sigma_{j+1}^X + (1-\gamma)\sigma_j^Y\sigma_{j+1}^Y) - \sum_{j=1}^{L}\omega_j\sigma_j^Z \\ &= \mathcal{C}^*\tilde{M}\mathcal{C} + E_0 I \end{aligned}$$

Here $\mathcal{C} = (c_1, \ldots, c_L, c_1^*, \ldots, c_L^*)^t$, $\mathcal{C}^* = (c_1^*, \ldots, c_L^*, c_1, \ldots, c_L)$,

Block Anderson model: $\tilde{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$

$$A = \begin{pmatrix} \omega_1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \omega_L \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\gamma & & \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & -\gamma \\ & & \gamma & 0 \end{pmatrix}$$

To reduce this to a free Fermion system one needs

General Bogolubov transformations (with mixing of creation/annihilation operators, not conserving the vaccum):

$W \in \mathbb{C}^{2L\times 2L}$ is called a *Bogolubov matrix* if

$$W \text{ unitary and } WJW^t = J, \text{ where } J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

If $\mathcal{C}$ satisfies CAR, then

$$\mathcal{B} = W\mathcal{C} \text{ satisfies CAR} \iff W \text{ Bogolubov}$$

More details on anisotropic XY: See Hamza/Sims/St. 2012, Abdul-Rahman/St. 2015

Part 3:

Droplets in the Infinite XXZ Chain

Based on:
Starr 2001
Nachtergaele/Starr 2001
Nachtergaele/Spitzer/Starr 2007
Fischbacher 2013
Fischbacher/St. 2014

Infinite spin configurations:

$\mathcal{H} = \mathcal{H}_{\mathbb{Z}}$ Hilbert space with (formal) ONB

$$B := \{e_\alpha = \bigotimes_{j\in\mathbb{Z}} e_{\alpha_j} : \alpha \in \{0,1\}^{\mathbb{Z}}, \sum_j \alpha_j < \infty\}$$

Infinite spin configurations with **finitely many down-spins** (particles), for example

$$e_\alpha = |\ldots \uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow \ldots\rangle$$

Recall: $e_0 = \begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle$, $e_1 = \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle$

The infinite free XXZ chain:

$$H = H_{\mathbb{Z}} = \sum_{i \in \mathbb{Z}} h_{i,i+1}$$

$$\begin{aligned} h_{i,i+1} &= \frac{1}{4}(I - \sigma_i^Z \sigma_{i+1}^Z) - \frac{1}{4\Delta}(\sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y) \\ &= \frac{1}{2}(a_i^* a_i + a_{i+1}^* a_{i+1}) - \frac{1}{2\Delta}(a_i^* a_{i+1} + a_{i+1}^* a_i) - a_i^* a_i a_{i+1}^* a_{i+1} \end{aligned}$$

Note: $e_\alpha \in B \implies h_{i,i+1} e_\alpha \neq 0$ for only finitely many i. H maps span B to span B.
In fact: H is essentially self-adjoint and unbounded on span B.
Write: $H = \overline{H|_{\mathrm{span}\, B}}$.
Choose $\Delta > 0$ (ferromagnetic), non-degenerate ground state:

$$H| \ldots \uparrow\uparrow\uparrow\uparrow \ldots \rangle = 0$$

Particle number preservation:

$$\mathcal{H}^{(N)} := \operatorname{span}\{e_\alpha : \sum_j \alpha_j = N\}$$

is invariant under H for all $N = 0, 1, 2, \ldots$. Thus $H = \bigoplus_{N=0}^{\infty} H_N$ for the bounded self-adjoint operators $H_N = \overline{H|_{\mathcal{H}^{(N)}}}$.

More precisely: Consider Fermionic N-particle configurations:

$$\mathcal{X}_N := \{x \in \mathbb{Z}^N : x_1 < x_2 < \ldots < x_N\}$$

Identify

$$e_\alpha \cong \delta_x =: \phi_x,$$

the standard basis vector in $\ell^2(\mathcal{X}_N)$ such that $\{x_1, \ldots, x_N\} = \{j : \alpha_j = 1\}$ (positions of down-spins in e_α).

Explicit calculation: (Possible Exercise: Do this calculation!)
$H_0 = 0$ on $|\ldots\uparrow\uparrow\uparrow\uparrow\ldots\rangle$ and

$$H_N = \overline{H|_{\mathcal{H}^{(N)}}} \cong -\frac{1}{2\Delta} h_0^{(\mathcal{X}_N)} + W + N, \quad N \geq 1$$

Here $h_0^{(\mathcal{X}_N)}$ is the adjacency operator on $\ell^2(\mathcal{X}_N)$:

$$(h_0^{(\mathcal{X}_N)} f)(x) = \sum_{y \in \mathcal{X}_N, \|x-y\|_1 = 1} f(y)$$

and W the attractive next-neighbor interaction potential

$$W(x_1, \ldots, x_N) = -\#\{j : x_{j+1} = x_j + 1\} = -\sum_{1 \leq k < \ell \leq N} Q(|x_k - x_\ell|)$$

with $Q(1) = 1$, $Q(r) = 0$ for $r \neq 1$.

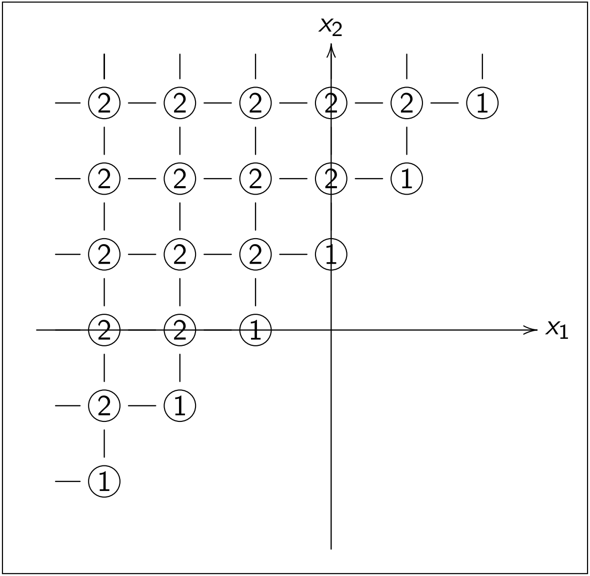


Figure: $H_2 = -\frac{1}{2\Delta} h_0^{(\mathcal{X}_2)} + W + 2$

- H_N is a bounded s.a. operator on $\ell^2(\mathcal{X}_N)$, $\|H_N\| \sim N$. May use $H := \bigoplus_N H_N$ to define infinite XXZ chain.
- XXZ chain is mapped to an interacting infinite Fermion system!
- The infinite XXZ chain is exactly solvable by the Bethe ansatz (Babbitt/Thomas/Gutkin 70's to 90's, Borodin/Corwin/Petrov/Sasamoto 2015)
- From now on assume

$$\Delta > 1 \quad \text{(Ising phase of XXZ)}$$

In this case we don't need the full Bethe ansatz (and its completeness) to understand the low energy spectrum.

Floquet-Bloch analysis of H_N:

H_N is invariant under translation of the "center of mass":

$$T_N(x_1, \ldots, x_N) = (x_1 + 1, \ldots, x_N + 1) \quad \text{on } \mathcal{X}_N$$

In particular, H_N is purely absolutely continuous.

See Figure for $N = 2$.

If Δ sufficiently large: Expect surface spectrum with generalized eigenfunctions concentrated along the "edge"

$$\mathcal{X}_{N,1} := \{(x_1, x_1 + 1, \ldots, x_1 + N - 1) : x_1 \in \mathbb{Z}\}$$

Explicit results for $N = 2$:

$$H_2 = \int_{[-\pi,\pi)}^{\oplus} H_2(\vartheta)\, d\vartheta$$

$$H_2(\vartheta) = \begin{pmatrix} 1 & -\frac{1+e^{i\vartheta}}{2\Delta} & & \\ -\frac{1+e^{-i\vartheta}}{2\Delta} & 2 & -\frac{1+e^{i\vartheta}}{2\Delta} & \\ & -\frac{1+e^{-i\vartheta}}{2\Delta} & 2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

$\Delta > 1$: Each $H_2(\vartheta)$ has a single eigenvalue $E_2(\vartheta) = 1 - \frac{1+\cos\vartheta}{2\Delta^2}$ below $\sigma_{ess}(H_2(\vartheta))$.

$$\begin{aligned} \sigma(H_2) &= \left[1 - \frac{1}{\Delta^2}, 1\right] \cup \left[2 - \frac{2}{\Delta}, 2 + \frac{2}{\Delta}\right] \\ &= \text{surface spectrum} \cup \text{bulk spectrum} \end{aligned}$$

$\Delta > 1$: There exists surface spectrum below bulk spectrum.

$\Delta > 2$: Gap between surface and bulk spectrum.

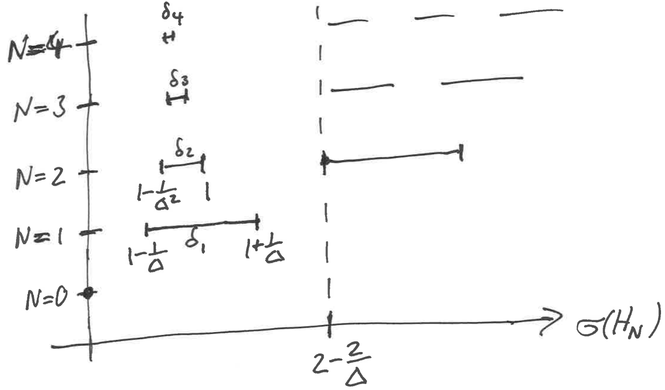
Generalized eigenfunctions for surface spectrum:

$$f_\vartheta(x_1, x_2) = e^{-i\vartheta x_1} \left(\frac{1 + e^{i\vartheta}}{2\Delta} \right)^{x_2 - x_1}$$

Thus: f_ϑ quasi-periodic along surface, exponentially decaying in bulk for $\Delta > 1$.

Possible Exercise: Verify these formulas!

$\sigma(H_N)$ for general N



- Surface spectral bands δ_N contract monotonically to $\delta_\infty = \{\sqrt{1 - \frac{1}{\Delta^2}}\}$
-
$$\sigma(H) = \bigcup_{N=0}^{\infty} \sigma(H_N) = \{0\} \cup [1 - \frac{1}{\Delta}, 1 + \frac{1}{\Delta}] \cup S$$
where $S \subset [2 - \frac{2}{\Delta}, \infty)$.
- The "droplet bands" $\delta_N \subset [1 - \frac{1}{\Delta}, 1 + \frac{1}{\Delta}]$ have generalized eigenfunctions which are concentrated on the edge $\mathcal{X}_{N,1}$ (with exponential decay in all bulk directions), corresponding to states which form a single droplet of downspins in a sea of upspins, i.e., linear combinations of

$$\ldots \uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow \ldots$$

- Note that $\mathcal{X}_{N,1}$ is "one-dimensional" within $\mathcal{X}_N$ for all N, also in the graph theoretical sense: $(x_1, x_2, \ldots, x_{N-1}, x_N) \in \mathcal{X}_{N,1}$ has only two next neighbors in $\mathcal{X}_N$:

$$(x_1 - 1, x_2, \ldots, x_{N-1}, x_N) \text{ and } (x_1, x_2, \ldots, x_{N-1}, x_N + 1)$$

- Expect: Adding disorder will localize the surface spectrum (compare: Jaksic/Last/Molchanov/Pastur $\sim$ 2000, and many others).
  Physically: Droplet can be viewed as a single quasi-particle which will get localized within the edge $\mathcal{X}_{N,1}$.
- Will this happen uniformly in N? Can the result be interpreted as MBL?

Part 4:

Localization of the Droplet Spectrum
in the Disordered XXZ Chain

Based on recent results of:

Beaud/Warzel 2017, Elgart/Klein/St. 2017

XXZ chain in random field:

$$H(\omega) = H_{\mathbb{Z}}(\omega) = \sum_{i\in\mathbb{Z}} h_{i,i+1} + \lambda \sum_i \omega_i \mathcal{N}_i$$

$$h_{i,i+1} = \frac{1}{4}(I - \sigma_i^Z \sigma_{i+1}^Z) - \frac{1}{4\Delta}(\sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y), \;\; \mathcal{N}_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i$$

Assume:

$$\omega = (\omega_i)_{i\in\mathbb{Z}} \text{ i.i.d.}, \;\; d\mu(\omega_i) = \rho(\omega_i)\, d\omega_i$$

$$\rho \text{ bounded}, \;\; \operatorname{supp} \rho = [0, \omega_{max}]$$

Disorder strength: $\lambda > 0$, Ising phase: $\Delta > 1$

Note: (i) $\lambda \sum_i \omega_i \mathcal{N}_i \geq 0$, (ii) $|\ldots \uparrow\uparrow\uparrow\uparrow \ldots\rangle$ remains ground state, $E_0 = 0$.

$H(\omega)$ particle number preserving, ergodic under shift, $\sigma(H_\omega) = \Sigma$ almost surely. In fact:

$$\sigma(H_\omega) = \{0\} \cup \left[1 - \frac{1}{\Delta}, \infty\right) \quad \text{a.s.}$$

Expect "localization" in droplet spectrum

$$I_1 = \left[1 - \frac{1}{\Delta}, 2 - \frac{2}{\Delta}\right)$$

Finite volume chain on $\mathcal{H}^{(L)} = \bigotimes_{i=-L}^{L} \mathbb{C}_i^2$:

$$H^{(L)}(\omega) = \sum_{i=-L}^{L-1} h_{i,i+1} + \lambda \sum_{i=-L}^{L} \omega_i \mathcal{N}_i + \beta(\mathcal{N}_{-L} + \mathcal{N}_L)$$

Assume: $\beta \geq \frac{1}{2}(1 - \frac{1}{\Delta})$ ("droplet b.c.", Nachtergaele/Starr)

Particle number conservation: $H^{(L)}(\omega) \cong \bigoplus_{N=0}^{\infty} H_N^{(L)}(\omega)$

$$H_N^{(L)}(\omega) = -\frac{1}{2\Delta} h_0^{(\mathcal{X}_N^{(L)})} + W + N + \lambda V_\omega + \left(\beta - \frac{1}{2}\right)\chi^{(L)}$$

on $\ell^2(\mathcal{X}_N^{(L)})$, where $\mathcal{X}_N^{(L)} = \{x \in \mathcal{X}_N : -L \leq x_1 < \ldots < x_N \leq L\}$,

$$V_\omega(x) = \sum_{j=1}^{N} \omega_{x_j} \quad N\text{-body Anderson random potential,}$$

$\chi^{(L)} = \chi_{-L} + \chi_L$ (indicator functions of $x_1 = -L$ and $x_N = L$)

A better way of writing $H_N^{(L)}(\omega)$:

(1) Replace $W(x) = -\#\{j : x_{j+1} = x_j + 1\}$ by

$\tilde{W}(x) = \#\{j : x_{j+1} > x_j + 1\} = W(x) + N =$ number of clusters in x

(2) Replace adjacency operator $h_0^{(\mathcal{X}_N^{(L)})}$ by graph Laplacian:

$$(\mathcal{L}_N^{(L)}\psi)(x) = \sum_{y \in \mathcal{X}_N^{(L)},\, \|x-y\|_1 = 1} (\psi(y) - \psi(x)), \quad \psi \in \ell^2(\mathcal{X}_N^{(L)})$$

Note: $h_0^{(\mathcal{X}_N^{(L)})} = \mathcal{L}_N^{(L)} + 2\tilde{W} - \chi^{(L)}$. Thus (1) and (2) imply

$$\begin{aligned} H_N^{(L)}(\omega) &= -\frac{1}{2\Delta} h_0^{(\mathcal{X}_N^{(L)})} + W + N + \lambda V_\omega + \Big(\beta - \frac{1}{2}\Big)\chi^{(L)} \\ &= -\frac{1}{2\Delta}\mathcal{L}_N^{(L)} + \Big(1 - \frac{1}{\Delta}\Big)\tilde{W} + \lambda V_\omega + \Big(\beta - \frac{1}{2}\big(1 - \frac{1}{\Delta}\big)\Big)\chi^{(L)} \\ &\geq 0 + 0 + 0 + 0 \end{aligned}$$

Simplicity of the spectrum:

Lemma: (Abdul-Rahman/St. 2015) Almost surely, all eigenvalues of $H^{(L)} = H^{(L)}(\omega)$ are simple. (Exercise: This holds for any s.a. operator $A + \sum_{i=-L}^{L} \omega_i \mathcal{N}_i$ on $\bigoplus_{j=-L}^{L} \mathbb{C}_j^2$ if the ω_i have a.c. distribution.)

Thus: May label eigenfunctions of $H^{(L)}$ by ψ_E, $E \in \sigma(H^L)$.

Definitions: (i) An observable $X \in B(\mathcal{H}^{(L)})$ is *supported on* $J \subset \{-L, \dots, L\}$ if $X \in B(\otimes_{i \in J} \mathbb{C}_i^2)$ acts trivially on all other spins. We write $J = \text{supp}(X)$.
(ii) The correlation of ψ w.r.t. observables X and Y is

$$R_{X,Y}(\psi) := |\langle \psi, XY\psi \rangle - \langle \psi, X\psi \rangle \langle \psi, Y\psi \rangle|.$$

(iii) Fix $\delta > 0$ (arbitrarily small) and let

$$I_{1,\delta} = \left[1 - \frac{1}{\Delta}, (2 - \delta)\left(1 - \frac{1}{\Delta}\right)\right]$$

Theorem 5 (Elgart/Klein/St. 2017)

If $\lambda\sqrt{\Delta - 1}$ is "sufficiently large", then there exist $C < \infty$ and $m > 0$ such that

$$\mathbb{E}\left(\sum_{E \in \sigma(H^{(L)}) \cap I_{1,\delta}} R_{X,Y}(\psi) \right) \leq C \|X\| \|Y\| e^{-m\, \mathrm{dist}(\mathrm{supp}(X), \mathrm{supp}(Y))}$$

uniformly in L, for all observables X, Y such that max supp$(X) <$ min supp(Y), or vice versa.

Remarks: (1) $\lambda\sqrt{\Delta - 1}$ sufficiently large means more precisely: There exists $K > 0$ (depending on δ and the distribution μ) such that for all $\Delta > 1$ and $\lambda > 0$ with

$$\lambda\sqrt{\Delta - 1}\, \min\{1, \Delta - 1\} \geq K$$

it holds that ....

(2) No dependence on sizes of $\mathrm{supp}(X)$ and $\mathrm{supp}(Y)$.

(3) Can take **sum over all correlations** in droplet spectrum.

(3) Result extends to *dynamical correlation*: For an interval $I \subset \mathbb{R}$ let $H_I^{(L)} = P_I H^{(L)}$ and $\tau_t^I(X) = e^{itH_I^{(L)}} X e^{-itH_I^{(L)}}$. Then

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \sum_{E\in\sigma(H^{(L)})\cap I_{1,\delta}} R_{\tau_t^{I_{1,\delta}}(X),Y}(\psi) \right) \leq \ldots$$

About the proof of Theorem 5:

Special case: $X = \mathcal{N}_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i$, $Y = \mathcal{N}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j$

$$\begin{aligned} R_{\mathcal{N}_i,\mathcal{N}_j}(\psi_E) &= |\langle \psi_E, \mathcal{N}_i \left(I - |\psi_E\rangle\langle\psi_E| \right) \mathcal{N}_j \psi_E \rangle| \\ &\leq \|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi_E\| \end{aligned}$$

Theorem 6 (Localization of **MB eigenfunction correlators**):

Under the above assumptions it holds that

$$\mathbb{E}\left(\sum_{E \in \sigma(H^{(L)}) \cap I_{1,\delta}} \|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi_E\| \right) \leq C e^{-m|i-j|}$$

for all $-L \leq i, j \leq L$, uniformly in L.

Fact: Theorem 6 is not just a special case of Theorem 5, but it can be shown to imply Theorem 5 for general local observables, including its generalization to dynamical correlations.

Can a bit more about the proof of this be said? At least for the case where both local observables are supported at only one site?

Had prepared to sketch proof on blackboard, but ran out of time. Contact me for details or see Section 3 in Elgart/Klein/St. 2017.

Reduction of Theorem 6 to N-body eigenfunction correlators:

Restriction of $\mathcal{N}_i$ to N-particle sector (L fixed):

$$Q_{i,N} = \mathcal{N}_i|_{\ell^2(\mathcal{X}_N^{(L)})} = \text{indicator function of } S_{i,N}^{(L)},$$

where

$$S_{i,N}^{(L)} = \{x \in \mathcal{X}_N^{(L)} : x_j = i \text{ for some } j \in \{1,\ldots,N\}\}$$

i.e., all lattice sites where random potential depends on ω_i.

Almost surely: Each ψ_E is non-degenerate and lies in a fixed N-particle sector. Thus

$$\|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi_E\| = \|Q_{i,N} \psi_E\| \|Q_{j,N} \psi_E\| = \|Q_{i,N} P_E Q_{j,N}\|_1$$

Here P_E is the spectral projection of $H_N^{(L)}$ onto E, and $\| \cdot \|_1$ the trace norm.

Thus, for any interval $I \subset \mathbb{R}$,

$$\sum_{E \in \sigma(H^{(L)}) \cap I} \|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi\| = \sum_{N=1}^{\infty} Q_N^{(L)}(i,j;I)$$

with the N-**body eigenfunction correlators**

$$Q_N^{(L)}(i,j;I) := \sum_{E \in \sigma(H_N^{(L)}) \cap I} \|Q_{i,N} P_E Q_{j,N}\|_1$$

Thus Theorem 6 is equivalent to

$$\sum_{N=1}^{\infty} \mathbb{E}\left(Q_N^{(L)}(i,j;I_{1,\delta})\right) \leq Ce^{-m|i-j|} \tag{1}$$

uniformly in L.

Remark: If one defines

$$\hat{Q}_N^{(L)}(i,j;I) := \sup\left\{\left\|Q_{i,N}\, g(H_N^{(L)})Q_{j,N}\right\|_1 :\ \operatorname{supp} g \subset I, |g| \leq 1\right\}$$

then $\hat{Q}_N^{(L)}(i,j;I) \leq Q_N^{(L)}(i,j;I)$, with equality for $N=1$ (where $Q_{i,N}$ is rank one), but not for $N \geq 2$. This and Theorem 6 imply with standard arguments that, almost surely, the infinite volume disordered XXZ chain $H(\omega)$ has **pure point spectrum** in $I_{1,\delta}$.

Bounding the N-body eigenfunction correlators:

Instead of a proof we make a series of remarks (some of them involving dry bones with little flesh):

Remark 1: We have reduced a result on many-body localization to a result on Anderson localization for an infinite system of N-body random Schrödinger operators. The challenge is that the latter has to be shown with bounds uniform in N (in fact, summable in N).

Remark 2: Bounds on the N-body eigenfunction correlators $Q_N^{(L)}$ are proven by (relatively traditional) Green's function methods. Getting the eigencorrelator bounds from Green's function bounds: See one (or both) of the preprints, which essentially follow known methods from Anderson localization theory. If one can prove the Green's function bounds uniform in N, this can be carried over to uniform eigencorrelator bounds.

Remark 3: Once one has uniformity in N, then summability in N in (1) is essentially due to a large deviations bound:

$$V_\omega(x) = \omega_{x_1} + \omega_{x_2} + \ldots + \omega_{x_N}$$

Thus

$$\mathbb{P}(V_\omega < 1) \leq Ce^{-cN}$$

Thus the appearance of droplet spectrum in the *random* XXZ chain is due to rare events: The sample size needs to be much larger than N. Approximately: In a sample of size L, the largest droplet is of size $\log L$. For "full" MBL this should grow linear in L.

This is physically not (yet) satisfying! (We have only scratched the surface.)

We discuss the Green's function bounds in infinite volume, so we can drop the extra index L. All results also hold in finite volume, with bounds uniform in L.

Remark 4: The Green's function bounds are proven separately for the edge $\mathcal{X}_{N,1}$ and for the bulk $\overline{\mathcal{X}}_{N,1} = \mathcal{X}_N \setminus \mathcal{X}_{N,1}$. These bounds are then related by Schur complementation with respect to

$$\ell^2(\mathcal{X}_{N,1}) \oplus \ell^2(\overline{\mathcal{X}}_{N,1})$$

Schur complementation is also used to analyze the Green's function along the edge.

Remark 5: The bulk Green's function is controlled by a Combes-Thomas bound:

Theorem 7 (Combes-Thomas bound)
Let $\Delta > 1$ and $\lambda > 0$ and let $\overline{H}_{N,1}$ denote the restriction of H_N to $\ell^2(\overline{\mathcal{X}}_{N,1})$. Then there exist constants $C = C(\Delta) < \infty$ and $\eta = \eta(\Delta) > 0$, independent of λ and N, such that

$$\|\chi_A(\overline{H}_{N,1} - E - i\epsilon)^{-1}\chi_B\| \leq Ce^{-\eta\, \mathrm{dist}_1(A,B)},$$

for all $N \in \mathbb{N}$, $E \in I_{1,\delta}$, $\epsilon \in \mathbb{R}$, and subsets A and B of $\overline{\mathcal{X}}_{N,1}$.

Here $\mathrm{dist}_1(A,B) = \inf_{x \in A, y \in B} \sum_j |x_j - y_j|$ is the 1-distance of A and B and χ_A, χ_B are indicator functions of A and B.

Recall: $H_N = -\frac{1}{2\Delta}\mathcal{L}_N + (1-\frac{1}{\Delta})\tilde{W} + \lambda V_\omega$, where

$$\tilde{W}(x) = \text{number of clusters in } (x_1, \ldots, x_N)$$

Thus: $\tilde{W} \geq 2$ on $\overline{\mathcal{X}}_{N,1}$ and $\overline{H}_{N,1} \geq 2(1-\frac{1}{\Delta})$, i.e., above the droplet spectrum $I_{1,\delta}$.
This is the classical situation where Combes-Thomas bounds for Schrödinger operators apply. However, the standard proof yields a dimension dependent decay rate η/N.

Key in proof of an N-independent bound:

$$\|\tilde{W}^{1/2}(\overline{H}_{N,1} - E - i\epsilon)^{-1}\tilde{W}^{1/2}\| \leq C(\delta, \Delta)$$

uniformly in $E \in I_{1,\delta}$, $\epsilon \in \mathbb{R}$ and $N \in \mathbb{N}$.

Remark 6: The edge Green's function is controlled by a fractional moment analysis:

Theorem 8: (Fractional moment bound on the edge)
If the parameters λ and Δ are in the region described in Theorem 5 (essentially: $\lambda\sqrt{\Delta - 1}$ sufficiently large), then there exist $C = C(\Delta)$ and $\xi = \xi(\Delta)$ such that

$$\mathbb{E}\left(\left|\langle \delta_u, (H_N - E - i\varepsilon)^{-1}\delta_v\rangle\right|^{1/2}\right) \leq \frac{C}{\sqrt{\lambda}} e^{-\xi\|u-v\|},$$

for all $N \in \mathbb{N}$, $E \in I_{1,\delta}$, $\epsilon > 0$, and $u, v \in \mathcal{X}_{N,1}$.

Here $\|u - v\| = \max\{|u_i - v_i| : 1 \leq i \leq N\}$ is the ∞-distance.

Remark 7: Note the difference between the 1-distance and the ∞-distance. One can't get a good (i.e. N-independent) "global" Green's function bound. The latter would essentially have to use the ∞-distance, which is much worse than the 1-distance for large N.
Instead, one works with a combination of Theorems 7 and 8 in getting the eigencorrelator bounds from the Green's function bounds.