Topics in Random Operator Theory

Simone Warzel

Zentrum Mathematik

TECHNISCHE UNIVERSITÄT MÜNCHEN

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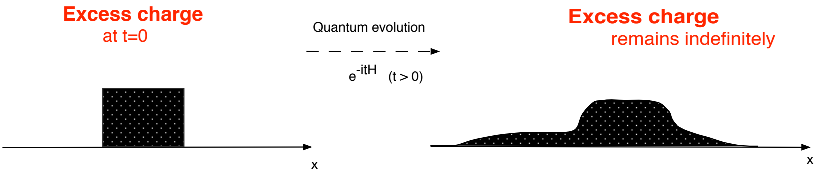
Many-body localization (MBL)

Physics phenomenon: disordered quantum systems may exhibit phases in which the equidistribution postulate is violated, i.e.

'typical stationary states are spread uniformly over the energy shell'

(ergodic hypothesis)



Prototype: *Non-interacting* quantum particle in a random potential

Anderson Localization

Challenge: Definition, existence & description of many-body localization for **interacting systems.**

Plan

Part I Localization for non-interacting particles in a nutshell

- The phase diagram
- Useful quantities
- Proof of localization at high disorder

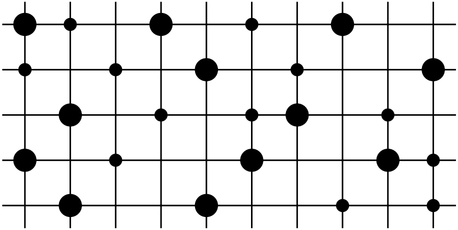
Part II Tiny steps towards understanding MBL

- Short overview over the mathematical literature
- Sketch of localization proof for 2 interacting particles
- Localization of a droplet in the XXZ spin chain
- Some simple consequences

Anderson model:

Single quantum particle in random potential on $\mathbb{G}$:

$$H = T + \lambda V , \qquad \text{in} \quad \ell^2(\mathbb{G})$$



- $(T\psi)(x) := \sum_{y:d(x,y)=1} \psi(y)$
- $\{V(x)\}_{x\in\mathbb{G}}$ iid random variables. $\qquad \mathbb{P}(V(x) \in dv) = \varrho(v)dv, \;\; \varrho \in L^\infty.$
- Disorder strength: $\quad \lambda > 0.$

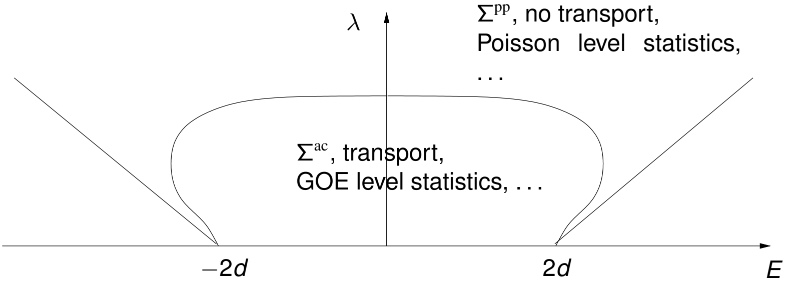
Expected phase diagram for $\mathbb{G} = \mathbb{Z}^d$:

Consequence of ergodicity: There are $\Sigma^\# \subset \mathbb{R}$ such that almost surely

$$\sigma_\#(H) = \Sigma^\#, \qquad \# = \cdot, \text{ac}, \text{pp}, \text{sc}$$

Dimensions d $=$ 1 or d $=$ 2: For any $\lambda > 0$: $\Sigma^{\text{sc}} = \Sigma^{\text{ac}} = \emptyset$

Dimension d $\geq$ 3: e.g. $\varrho(v) = 1_{[-\frac{1}{2}, \frac{1}{2}]}(v)$



Quantities capturing the transition

1. **Resolvent** and **Green function** at $z \in \mathbb{C} \backslash \sigma(H)$

$$G(x, y; z) = \langle \delta_x, (H - z)^{-1} \delta_y \rangle = \int (E - z)^{-1} \mu_{x,y}(dE)$$

2. **Eigenfunction correlator** aka **total variation of spectral measure**

$$Q(x, y; I) = |\mu_{x,y}|(I) = \sum_{E \in \sigma(H) \cap I} |\psi_E(x)| \, |\psi_E(y)|$$

where the last equality requires simple spectrum.

Some relations

- **Transport:** $\sup_{t \in \mathbb{R}} \left| \langle \delta_x, e^{-itH} P_I(H) \delta_y \rangle \right| \leq Q(x, y; I)$

  Spectral information then available through the RAGE theorem.
- For $|\mathbb{G}| < \infty$: $Q(x, y; I) = \lim_{s \uparrow 1} \frac{1 - s}{2} \int_I |G(x, y; E)|^s \, dE$

Anderson's argument for localization at high disorder

Iterating the resolvent equation: $$\frac{1}{H-z} = \frac{1}{\lambda V - z} - \frac{1}{\lambda V - z} T \frac{1}{H-z}$$

leads for $\operatorname{Im} z > \|T\|$ to a convergent path expansion:

$$G(x,y;z) = \sum_{\gamma : x \mapsto y} (-1)^{|\gamma|} \, T(\gamma(0),\gamma(1)) \, T(\gamma(1),\gamma(2)) \cdots T(\gamma(|\gamma|-1),\gamma(|\gamma|)) \prod_{k=0}^{|\gamma|} \frac{1}{V(\gamma(k))-z}$$

Intuition: The $x \in \mathbb{G}$ for which $|\lambda V(x) - z| \geq \deg_{\mathbb{G}}$ are spatially rare.
Hence most paths collect an exponential decay.

Problem: For $z \in \lambda \operatorname{supp} \varrho$ exceptional $x \in \mathbb{G}$ exist for which arbitrarily large amplification occurs in this expansion exists. Their contribution is however suppressed probabilstically, i.e., for any $t > 0$:

$$\sup_{\alpha \in \mathbb{C}} \mathbb{P}\left(\frac{1}{|\lambda V(x) - \alpha|} > t \right) \leq \frac{2\|\varrho\|_\infty}{t\lambda} .$$

Hence, for any $s \in (0,1)$:
$$\sup_{\alpha \in \mathbb{C}} \int \frac{\varrho(v) dv}{|\lambda v - \alpha|^s} \leq \frac{(2\|\varrho\|_\infty)^s}{(1-s)\,\lambda^s} =: \frac{C_s}{\lambda^s} < \infty .$$

Technique: (cf. Anderson '58)

Feenberg expansion, i.e., loop-erased version: (wlog $x \neq y$)

$$G(x,y;z) = \sum_{\hat\gamma : x \mapsto y}^{(SAW)} (-1)^{|\hat\gamma|}\, T(\hat\gamma(0), \hat\gamma(1))\ T(\hat\gamma(1), \hat\gamma(2))$$
$$\cdots T(\hat\gamma(|\hat\gamma|-1), \hat\gamma(|\hat\gamma|)) \prod_{k=0}^{|\hat\gamma|} \langle \delta_{\hat\gamma(k)}, \frac{1}{H_{\{\hat\gamma,k\}^c} - z}\, , \delta_{\hat\gamma(k)} \rangle ,$$
$$= -\, G(x,x;z) \sum_{v : d(x,v) = 1} T(x,v)\, G_{\{x\}^c}(v,y;z) .$$

where G_B is the Green function restricted to $B \subset \mathbb{G}$.

Proof of localization at high disorder

$$G(x,y;z) = -G(x,x;z) \sum_{v:d(x,v)=1} T(x,v)\, G_{\{x\}^c}(v,y;z)\,.$$

Taking $s \in (0,1)$ moments and (conditional) expectation values:

$$\begin{aligned}
\mathbb{E}\left[|G(x,y;z)|^s\right] &\leq \sum_{v:d(x,v)=1} \mathbb{E}\left[|G(x,x;z)|^s\, |G^{(x)}(v,y;z)|^s\right] \\
&= \sum_{v:d(x,v)=1} \mathbb{E}\left[\mathbb{E}_x\left[|G(x,x;z)|^s\right]\, |G^{(x)}(v,y;z)|^s\right] \\
&\leq \frac{C_s}{\lambda^s} \sum_{v:d(x,v)=1} \mathbb{E}\left[|G^{(x)}(v,y;z)|^s\right]\,.
\end{aligned}$$

Use Feshbach-Krein-Schur formula, i.e. $G(x,x;z) = (\lambda V(x) - \alpha(z))^{-1}$ with some $\alpha(z) \in \mathbb{C}$.

Under the condition $\boxed{e^{-\mu_s} := \deg_{\mathbb{G}} \frac{C_s}{\lambda^s} < 1\,,}$ iteration yields:

$$\mathbb{E}\left[|G(x,y;z)|^s\right] \leq \frac{C_s}{\lambda^s} e^{-\mu_s d(x,y)}\,. \qquad \square$$

Recall: Feshbach-Krein-Schur formula

Theorem

Let H be a self-adjoint operator in some Hilbert space $\mathcal{H}$ and let P be an orthogonal projection onto a closed subspace on which we define

$$K(z) := P\,(H-z)^{-1}\,P\,, \qquad z \in \mathbb{C}\backslash\mathbb{R}\,.$$

Then for any $z \in \mathbb{C}\backslash\mathbb{R}$ and any bounded self-adjoint operator of the form $A = PAP$,

- *the operator $1 + A\,K(z)$ is invertible on $P\mathcal{H}$,*
- *on $P\mathcal{H}$ one has the identity*

$$\boxed{(H+A-z)^{-1}\,P = (H-z)^{-1}\,P\,[1 + A\,K(z)]_P^{-1}\,P}\,.$$

Exercise: *Prove this (for matrices) and apply it to the case $P = |\delta_x\rangle\langle\delta_x|$!*

Localization at high disorder

Theorem

If $\lambda > (C_s \deg_{\mathbb{G}})^{1/s}$ *for some* $s \in (0,1)$, *then there is* $\mu \in (0,\infty)$ *such that for any bounded Borel* $I \subset \mathbb{R}$ *and all* $x, y \in \mathbb{G}$*:*

$$\mathbb{E}\left[Q(x,y;I)\right] \leq A_I \, e^{-\mu d(x,y)}$$

at some $A_I < \infty$.

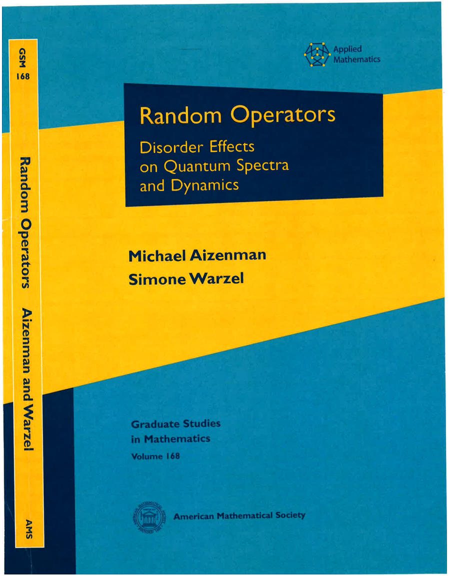
- For $d = 1$ proven by Kunz/Souillard '82
- For arbitrary d proven by Aizenman '94
- Other trailblazing works on Anderson localization:

Goldsheid/Molchanov/Pastur '73 $(d = 1)$

Fröhlich/Spencer '83 (Multiscale Analysis $d \geq 1$)

Aizenman/Molchanov '92 (Fractional Moments $d \geq 1$)

More information



Proposal by Basko/Aleiner/Altshuler '06:

Starting from a non-interacting system of n particles on $\Lambda \subset \mathbb{Z}^d$ in the completely localized phase, turning on weak interactions amounts to an effective sparse hopping on the graph of particle configurations

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Lambda^n$$

Essentially the same argument as in the one-particle case, is proposed to lead to exponential decay of the **many-particle Green function** $G_\Lambda^{(n)}(\mathbf{x}, \mathbf{y}, z)$.

Problem:

Control of resonant spots $\mathbf{x}$ is much more complicated due much less randomness in configuration space!

Some selected mathematical efforts

1. Systems of a **fixed but arbitrary number of particles**:
   - Aizenman/W. '09, Chulaevsky/Suhov '09–'14
   - Droplet spectrum of the **XXZ spin chain**

     Beaud/W. arXiv:1703.02465

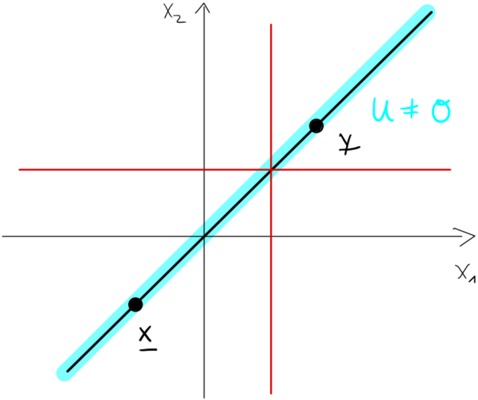
     Elgart/Klein/Stolz arXiv:1703.07483
2. Localization in **integrable models**
   - for **XY spin chain** Hamza/Sims/Stolz '08, Sims/W. 16, ... Abdul-Rahman/Nachtergaele/Sims/Stolz '16
   - for **Tonks-Girarndeau gas** Seiringer/W. '16
3. **Decay of correlations of all states in certain one-dimensional spin chain**

   Imbrie '14
4. **Ground-state localisation of weakly interacting fermions**
   - for **Aubry-André potential** Mastropietro '16
   - within the **Hartee-Fock theory** Ducatez '16

Example: Localization for 2 hard-core particles on $\Lambda \subset \mathbb{Z}$

Consider $H = \sum_j (T + \lambda V)_j + U$ in $\ell^2(\chi^2)$ e.g. with nn interaction U.

Configuration space: $\chi^2 := \{\mathbf{x} = (x_1, x_2) \in \Lambda^2 \mid x_1 < x_2\}$



Clustered configurations: $\mathcal{C} = \{\mathbf{x} \mid x_2 = x_1 + 1\}$

Sketch of localization proof for $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ with $x_1 < y_1$

Similarly to the one-particle case a 'last exit' resolvent expansion into $\Lambda_{\mathbf{x}} = \Lambda \cap [x_1 + 1, \infty)$ leads to:

$$|G_\Lambda(\mathbf{x}, \mathbf{y}; z)| \leq |G_\Lambda(\mathbf{x}, \mathbf{u}(\mathbf{v}_0); z)||G_{\Lambda_{\mathbf{x}}}(\mathbf{v}_0, \mathbf{y}; z)| + \sum_{\substack{\mathbf{v} \notin \mathcal{C} \\ v_1 = x_1 + 1}} |G_\Lambda(\mathbf{x}, \mathbf{u}(\mathbf{v}); z)||G_{\Lambda_{\mathbf{x}}}(\mathbf{v}, \mathbf{y}; z)|,$$

where $\mathbf{v}_0 = (x_1 + 1, x_2 + 1) \in \mathcal{C}$, and $\mathbf{u}(\mathbf{v})$ is the unique neighboring configuration of $\mathbf{v}$ with particle outside $\Lambda_{\mathbf{x}}$.

- Use: $\mathbb{E}_\xi\left[|G_\Lambda(\mathbf{x}, \mathbf{y}; z)|^s\right] \leq \frac{C_s}{\lambda^s}$ for any configuration with $\xi \in \mathbf{x}$ and $\xi \in \mathbf{y}$.
- Expand the second factor in terms of the Green function $G^{(2)}$ of the operator projected on non-clustered configurations:

$$|G_{\Lambda_{\mathbf{x}}}(\mathbf{v}, \mathbf{y}; z)| \leq \sum_{\substack{\mathbf{w} \in \mathcal{X}_{\mathbf{x}} \setminus \mathcal{C}_{\mathbf{x}} \\ \mathbf{z} \in \mathcal{C}_{\mathbf{x}}, d(\mathbf{w}, \mathbf{z}) = 1}} |G^{(2)}_{\Lambda_{\mathbf{x}}}(\mathbf{v}, \mathbf{w}; z)||G_{\Lambda_{\mathbf{x}}}(\mathbf{z}, \mathbf{y}; z)|$$

The proof may proceed supposing

$$|G^{(2)}_{\Lambda_{\mathbf{x}}}(\mathbf{v}, \mathbf{w}; z)| \leq C\, e^{-\mu_T d(\mathbf{v}, \mathbf{w})}$$

to conclude by iteration for $\mathbf{x}, \mathbf{y} \in \mathcal{C}$: (*Exercise: show this!*)

$$\mathbb{E}\left[|G_\Lambda(\mathbf{x}, \mathbf{y}; z)|^s\right] \leq \frac{\widehat{C}_s}{\lambda^s} e^{-\mu_s |x_1 - y_1|}$$

Sketch of localization proof for $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ with $x_1 < y_1$

Two cases for which

$$|G^{(2)}_{\Lambda_\mathbf{x}}(\mathbf{v}, \mathbf{w}; z)| \leq C\, e^{-\mu_T d(\mathbf{v},\mathbf{w})}$$

holds:

1. $G^{(2)}_{\Lambda_\mathbf{x}}$ is essentially **non-interacting** for which the above was already established at least in expectation.

   Aizenman/W. '09, Chulaevsky/Suhov '09–'14

2. assume **attractive interaction** and the fact that $\mathrm{Re}\, z < E_{\text{Cluster Break-up}}$ such that the above follows by a **Combes-Thomas bound**.

   Beaud/W. '17, Elgart/Klein/Stolz '17

XXZ spin chain in random field

Spins $\frac{1}{2}$ on $\Lambda := [1, L] \cap \mathbb{Z}$: $\mathcal{H}_{\mathrm{L}}^{\mathrm{XXZ}} = \bigotimes_{k=1}^{L} \mathbb{C}_k^2$

$$H_{\mathrm{L}}^{\mathrm{XXZ}} := -\sum_{k=1}^{L-1} \left[\frac{1}{\Delta} \left(S_k^x \otimes S_{k+1}^x + S_k^y \otimes S_{k+1}^y \right) + \left(S_k^z \otimes S_{k+1}^z - \tfrac{1}{4} \mathbb{1}_k \otimes \mathbb{1}_{k+1} \right) \right]$$

$$+ \tfrac{1}{2} \left(\mathbb{1} - S_1^z - S_{\mathrm{L}}^z \right) + \frac{\lambda}{\Delta} \sum_{k=1}^{N} \omega(k) \left(\tfrac{1}{2} \mathbb{1} - S_k^z \right),$$

- droplet BC
- anisotropy parameter $\Delta > 0$. Here: **Ising phase $\Delta > 1$**
- $\{\omega(k)\}$ iid random variables,

  $\mathbb{P}(\omega(k) \in dv) = \varrho(v)\, dv$ with $\varrho \in L^\infty$ and $\mathrm{supp}\, \varrho \subset [0, \omega_{\max}]$.

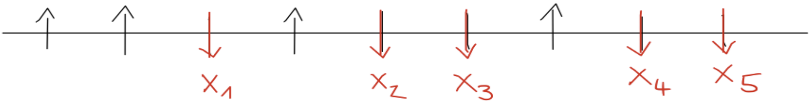
  disorder parameter $\lambda > 0$.

- conservation law: $\left[H_{\mathrm{L}}^{\mathrm{XXZ}}, \sum_{k=1}^{L} S_k^z \right] = 0$

Unitary equivalence of XXZ to hard-core attractive particles

Spin configuration with **fixed number of n down (z) spins** is identified with **ordered particle configurations** on Λ, i.e.,

$$\mathcal{X}^n := \{\, \mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in \Lambda^n \ :\ x_1 < x_2 < \ldots < x_n \,\}$$



Unitary equivalence:

$$\mathcal{U} : \quad \mathcal{H}_L^{XXZ} \longrightarrow \bigoplus_{n=0}^{L} \ell^2(\mathcal{X}^n)$$

$$2\Delta\, H_L^{XXZ} \longrightarrow H := -A + \lambda V + 2\Delta\, U\,,$$

- Hopping of particle configuartions $\mathbf{x}, \mathbf{y} \in \mathcal{X}^n$ at **distance** $d(\mathbf{x},\mathbf{y}) := \sum_{j=1}^{n} |x_j - y_j|$

$$A\delta_{\mathbf{x}} := \sum_{\substack{\mathbf{y}\in\mathcal{X}^n \\ d(\mathbf{x},\mathbf{y})=1}} \delta_{\mathbf{y}}\,,$$

- Random potential:

$$V\delta_{\mathbf{x}} = \left(\sum_{j=1}^{n} \omega(x_j) \right) \delta_{\mathbf{x}}.$$

Unitary equivalence of XXZ to hard-core attractive particles

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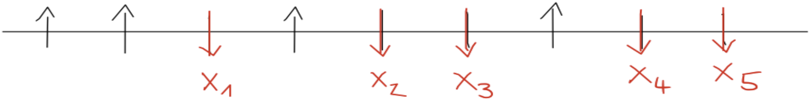

Unitary equivalence:

$$\mathcal{U} : \quad \mathcal{H}_{\mathrm{L}}^{\mathrm{XXZ}} \longrightarrow \bigoplus_{n=0}^{L} \ell^2(\mathcal{X}^n)$$

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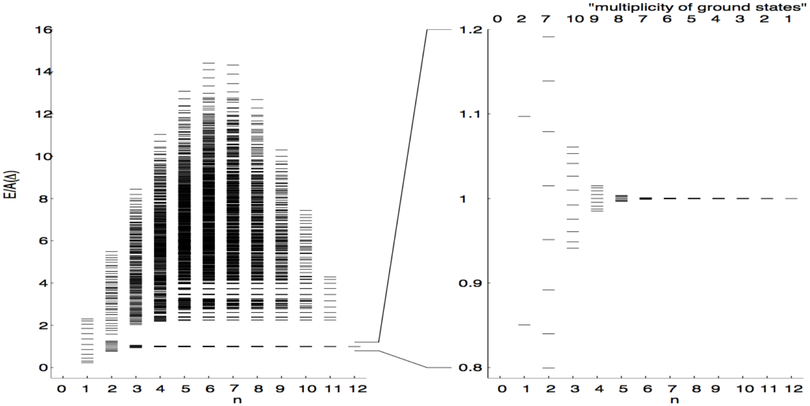
Cluster decomposition: $\mathcal{X}^n = \bigcup_{k=1}^{n} \mathcal{C}^{(k)}$

- Interaction for k-cluster configurations $\mathbf{x} \in \mathcal{C}^{(k)}$: $U\delta_{\mathbf{x}} := k\delta_{\mathbf{x}}.$

Spectrum in Ising phase $\Delta > 1$ without disorder $\lambda = 0$ Nachtergaele/Starr '01



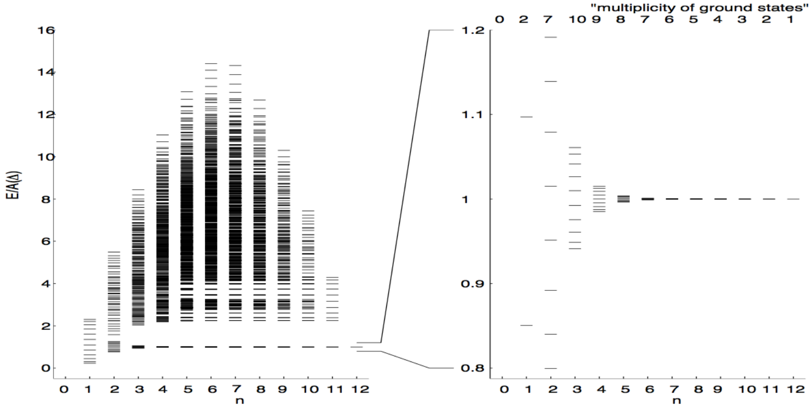
(Energy scale: $A(\Delta) := \frac{1}{2}\sqrt{1 - \Delta^{-2}}$, cf. Nachtergaele/Starr '01)

Droplet band for fixed n:

$$\Delta(n) := 2\sqrt{\Delta^2 - 1}\left[\frac{\cosh(\rho_\Delta n) - 1}{\sinh(\rho_\Delta n)}, \frac{\cosh(\rho_\Delta n) - 1}{\sinh(\rho_\Delta n)}\right] \subset \left[2(\Delta - 1), 2(\Delta + 1)\right]$$

where $\rho_\Delta := \ln(\Delta + \sqrt{\Delta^2 - 1})$

Spectrum in Ising phase $\Delta > 1$ without disorder $\lambda = 0$ Nachtergaele/Starr '01



(Energy scale: $A(\Delta) := \frac{1}{2}\sqrt{1-\Delta^{-2}}$, cf. Nachtergaele/Starr '01)

Let $Q^{(k)}$ stand for the orthogonal projection onto the subspace $\bigoplus_{j=k}^{\infty} \ell^2(\mathcal{C}^{(j)})$ of **at least** k **clusters**, then: $Q^{(k)} H Q^{(k)} \geqslant 2k(\Delta - 1)$.

Main quantity of interest and basic spectral information

Eigenfunctions correlator $I \subset \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{X}^n$

$$Q^{(n)}(\mathbf{x}, \mathbf{y}, I) := \sum_{E \in \sigma(H) \cap I} \left| \langle \delta_{\mathbf{x}}, P_{\{E\}}(H) \delta_{\mathbf{y}} \rangle \right| ,$$

- Many-particle density of states in I corresponding to $\mathbf{x} \in \mathcal{X}^n$:

$$\mathbb{E}\left[Q^{(n)}(\mathbf{x}, \mathbf{x}, I)\right] \leq \inf_{t>0} e^{t \sup I} \, \mathbb{E}\left[e^{-t\lambda V(\mathbf{x})}\right] \leq C(I) \, e^{-c\lambda n} .$$

exponential supression!

Main quantity of interest and basic spectral information

The ground-state energy of H on an interval $\Lambda \subset \mathbb{Z}$

$$\inf \sigma\left(H_\Lambda\right) \geqslant 2(\Delta-1) + \min\left\{2(\Delta-1), \lambda V^{\mathcal{C}}_{\min}\right\},$$

with $V^{\mathcal{C}}_{\min} := \min_{\mathbf{x}\in\mathcal{C}} \sum_{i=1}^{n} \omega(x_i)$.

- By a Chernoff bound

$$\mathbb{P}\left(\lambda V^{\mathcal{C}}_{\min} \leqslant E\right) \leqslant (|\Lambda| - n + 1) \inf_{t>0} e^{tE} \left(\int e^{-t\lambda\omega} \varrho(\omega) \mathrm{d}\omega\right)^n,$$

vanishes in the limit $|\Lambda| \to \infty$ if n is proportional to $|\Lambda|$.

no statement about positive density can be made!

Main result

Theorem (Beaud/W. '17 – in slightly different form: Elgart/Klein/Stolz '17)

Let $\Delta > 1$ and $\mu_T > 0$ be such that

$$E(\Delta, \mu_T) := 4\Delta - 12e^{\mu_T} > 0\,,$$

and let $I \subset [0, E(\Delta, \mu_T))$ be a compact interval and $\mu \in (0, \mu_T)$. There exist constants $\lambda_0, c, C \in (0, \infty)$ such that for all $n \geq 2$, Λ, all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}^n$, and all $\lambda > \lambda_0$:

$$\boxed{\mathbb{E}\big[|Q^{(n)}(\boldsymbol{x}, \boldsymbol{y}, I)|\big] \leqslant C\, e^{-c\lambda n}\, F_{\mu/2}(\boldsymbol{x}, \boldsymbol{y})}$$

where $F_\mu(\boldsymbol{x}, \boldsymbol{y}) :=$

$$\begin{cases} \exp\big(-\mu|x_1 - y_1|\big) & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}, \\ \sum_{\boldsymbol{w} \in \mathcal{C}} \exp\big(-\mu\,(d(\boldsymbol{x}, \boldsymbol{w}) + |w_1 - y_1|)\big) & \text{if } \boldsymbol{x} \notin \mathcal{C} \text{ and } \boldsymbol{y} \in \mathcal{C}, \\ \sum_{\boldsymbol{w}, \boldsymbol{v} \in \mathcal{C}} \exp\big(-\mu\,(d(\boldsymbol{x}, \boldsymbol{w}) + d(\boldsymbol{v}, \boldsymbol{y}) + |w_1 - v_1|)\big) & \text{if } \boldsymbol{x}, \boldsymbol{y} \notin \mathcal{C}. \end{cases}$$

Here $\mathcal{C} \equiv \mathcal{C}^{(1)}$.

Main result

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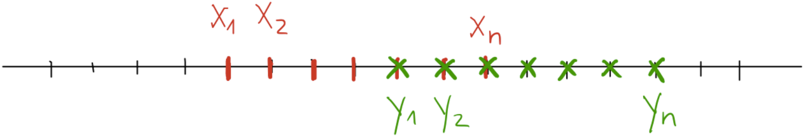
$$\boxed{\mathbb{E}\big[|Q^{(n)}(\mathbf{x}, \mathbf{y}, I)|\big] \leqslant C\, e^{-c\lambda n}\, F_{\mu/2}(\mathbf{x}, \mathbf{y})}$$

- Decay of clustered configurations $\mathbf{x}, \mathbf{y} \in \mathcal{C}$:



$$F_\mu(\mathbf{x}, \mathbf{y}) = \exp\big(-\mu|x_1 - y_1|\big)$$

Main result

Theorem (Beaud/W. '17 – in slightly different form: Elgart/Klein/Stolz '17)

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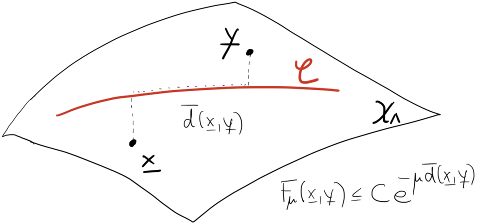
$$\boxed{\mathbb{E}\big[|Q^{(n)}(\boldsymbol{x}, \boldsymbol{y}, I)|\big] \leqslant C\, e^{-c\lambda n}\, F_{\mu/2}(\boldsymbol{x}, \boldsymbol{y})}$$

- Decay of general configurations:



Strong summability properties of F

Lemma

Let $U, V \subset \Lambda$ be two connected subsets with $\sup U < \inf V$ and fix $\mu > 0$. Then, there is $C_\mu \in (0, \infty)$ such that

$$\sum_{\substack{\mathbf{x} \in \mathcal{X}^n \\ \mathbf{x} \cap U \neq \emptyset}} \sum_{\substack{\mathbf{y} \in \mathcal{X}^n \\ \mathbf{y} \cap V \neq \emptyset}} F_\mu(\mathbf{x}, \mathbf{y}) \leqslant C_\mu \, (n+1) \, .$$

Immediate consequences

Density matrix of $\psi \in \ell^2(\mathcal{X}^n)$: $\gamma_\psi(\xi,\eta) := \sum_{\substack{\mathbf{x}\in\mathcal{X}^n \\ \xi\in\mathbf{x}}} \sum_{\substack{\mathbf{y}\in\mathcal{X}^n \\ \eta\in\mathbf{y}}} \overline{\psi(\mathbf{x})}\psi(\mathbf{y}).$

Corollary (Decay of time-dependent one-particle density matrix)

In the situation of the theorem, there is some $\nu > 0$ such that for any n, L, and any eigenstates $\psi_E \in \ell^2(\mathcal{X}^n)$ and $\xi,\eta \in \Lambda$:

$$\mathbb{E}\left[\sum_{E\in I\cap\sigma(H^{(n)})} |\gamma_{\psi_E}(\xi,\eta)|\right] \leq C\, e^{-c\lambda n}\, e^{-\nu|\xi-\eta|}\,.$$

- Exponential clustering
- The spectrum is almost surely simple — *cf.* Abduhl-Rahman/Stolz '16

Let ρ be any state, e.g. $\varrho = |\psi_E\rangle\langle\psi_E|$, and pick $U \subset \Lambda$

$$\rho_U := \mathrm{Tr}_{U^c}\, \rho\,, \qquad \text{acts on} \qquad \bigoplus_{m=0}^{n} \ell^2(\mathcal{X}_U^m).$$

be the **reduced state** associated with $U \subset \Lambda$. Its **Rényi entropy** is:

$$S_\alpha(\rho_U) := \frac{1}{1-\alpha} \ln \mathrm{Tr}\big(\rho_U^\alpha\big)\,, \qquad \alpha \in [0,\infty]\,.$$

Recall:

- $\alpha = 1$ von Neumann entropy.
- Monotonicity: $\alpha \le \beta$ implies $S_\alpha \ge S_\beta \ge 0$.

Immediate consequences *(cont.)*

Let ρ be any state, e.g. $\varrho = |\psi_E\rangle\langle\psi_E|$, and pick $U \subset \Lambda$

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be the **reduced state** associated with $U \subset \Lambda$. Its **Rényi entropy** is:

$$\boxed{S_\alpha(\rho_U) := \frac{1}{1-\alpha} \ln \mathrm{Tr}\left(\rho_U^\alpha\right), \qquad \alpha \in [0,\infty]\,.}$$

Corollary (Area law for entropy)

In the situation of the Theorem, for any $\alpha \in (0,1)$ there is $C_\alpha \in (0,\infty)$ such that for any n, L and $U \subset \Lambda$:

$$\mathbb{E}\left[e^{(1-\alpha)\, S_\alpha([|\psi\rangle\langle\psi|]_U)}\right] \leq C_\alpha\,.$$

for all $\psi = P_I(H)\psi \in \ell^2(\mathcal{X}^n)$.

- Logarithmic behavior in case of no disorder
- In agreement with numerical findings in Znidaric/Prosen/ Prelovsek '08 and Bauer/Nayak '13

Proof of area law for $\rho = |\psi\rangle\langle\psi|$ with $\psi \in \ell^2(\mathcal{X}^n)$ and $0 < \alpha < 1$

The reduced density matrix ρ_U decomposes into the case of having $m = 0, 1, \ldots, \min\{|U|, n\}$ particles on U:

$$\rho_U = \bigoplus_{m=0}^{\min\{|U|,n\}} \rho_U^{(m)}, \qquad \rho_U^{(m)}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{z} \in \mathcal{X}_{U^c}^{n-m}} \overline{\psi^n(\{\mathbf{x}, \mathbf{z}\})} \psi^n(\{\mathbf{y}, \mathbf{z}\}) .$$

Thus if $|U| \geq n$:

$$\mathrm{Tr}(\rho_U^\alpha) = \sum_{m=0}^{n} \mathrm{Tr}((\rho_U^{(m)})^\alpha) \leq 2 + \sum_{m=1}^{n-1} \sum_{\mathbf{x} \in \mathcal{X}_U^m} \langle \delta_\mathbf{x}, \rho_U^\alpha \delta_\mathbf{x} \rangle \leq 2 + \sum_{m=1}^{n-1} \sum_{\mathbf{x} \in \mathcal{X}_U^m} \langle \delta_\mathbf{x}, \rho_U \delta_\mathbf{x} \rangle^\alpha$$

$$\leq 2 + \sum_{m=1}^{n-1} \sum_{\mathbf{x} \in \mathcal{X}_U^m} \sum_{\mathbf{z} \in \mathcal{X}_{U^c}^{n-m}} \left| \psi^n(\{\mathbf{x}, \mathbf{z}\}) \right|^{2\alpha}$$

$$\leq 2 + \sum_{m=1}^{n-1} \sum_{\mathbf{x} \in \mathcal{X}_U^m} \sum_{\mathbf{z} \in \mathcal{X}_{U^c}^{n-m}} \sum_{\mathbf{y} \in \mathcal{X}^n} Q^{(n)}((\mathbf{x}, \mathbf{z}), \mathbf{y}, I)^\alpha$$

Take expection values:

$$\mathbb{E}\left[\mathrm{Tr}(\rho_U^\alpha)\right] \leq 2 + \sum_{m=1}^{n-1} \sum_{\mathbf{x} \in \mathcal{X}_U^m} \sum_{\mathbf{z} \in \mathcal{X}_{U^c}^{n-m}} \sum_{\mathbf{y} \in \mathcal{X}^n} C\, e^{-c\lambda n} F_{\alpha\mu/2}((\mathbf{x}, \mathbf{z}), \mathbf{y}) . \quad \square$$

Proof of area law for $\rho = |\psi\rangle\langle\psi|$ with $\psi \in \ell^2(\mathcal{X}^n)$ and $0 < \alpha < 1$

The reduced density matrix ρ_U decomposes into the case of having $m = 0, 1, \ldots, \min\{|U|, n\}$ particles on U:

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Thus if $|U| \geq n$: Take expection values:

$$\mathbb{E}\left[\mathrm{Tr}(\rho_U^\alpha)\right] \leq 2 + \sum_{m=1}^{n-1} \sum_{\mathbf{x} \in \mathcal{X}_U^m} \sum_{\mathbf{z} \in \mathcal{X}_{U^c}^{n-m}} \sum_{\mathbf{y} \in \mathcal{X}^n} C\, e^{-c\lambda n} F_{\alpha\mu/2}((\mathbf{x}, \mathbf{z}), \mathbf{y}). \qquad \square$$

Use

Lemma

Let $U \subset \Lambda$ be a connected strict subset, $U^c := \Lambda \setminus U$ and $\mu > 0$. Then, there exists a second-order polynomial $C(n)$ in n (depending on μ) such that

$$\sum_{\substack{\boldsymbol{x} \in \mathcal{X}^n \\ \boldsymbol{x} \cap U \neq \emptyset \\ \boldsymbol{x} \cap U^c \neq \emptyset}} \sum_{\boldsymbol{y} \in \mathcal{X}^n} F_\mu(\boldsymbol{x}, \boldsymbol{y}) \leqslant C(n).$$